PART I: DO ALL 7 PROBLEMS. (50 pts)

1. (8 pts) The columns in the chart below contain lists of vectors in $\mathbb{R}^3$. The letters $s, t, u$ denote real variables that can take any values. In each of the 8 boxes, enter A for always, S for sometimes, or N for never. You need not give reasons.

<table>
<thead>
<tr>
<th>As $s, t,$ and $u$ vary, are the vectors</th>
<th>$(s, 0, 1)$, $(0, t, 1)$, $(0, 0, u)$, $(1, 1, 1)$</th>
<th>$(1, 0, 0)$, $(0, t, 0)$</th>
<th>$(1, 0, 0)$, $(0, t, 1)$</th>
<th>$(1, s, t)$, $(0, 1, u)$, $(0, 0, 1)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>linearly independent?</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Do they span $\mathbb{R}^3$?</td>
<td></td>
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</tbody>
</table>

2. (5 pts) Two of the eigenvalues of the $4 \times 4$ matrix below are $-1$ and $3$. What are the other two and why?

\[
\begin{pmatrix}
-1 & 0 & 0 & 0 \\
-21 & 1 & 0 & 2 \\
15 & -1 & 4 & 1 \\
-8 & -4 & 0 & 7
\end{pmatrix}
\]

3. (10 pts) Let $P$ be the subspace of $\mathbb{R}^4$ consisting of those vectors $(x_1, x_2, x_3, x_4)$ satisfying $x_1 + x_2 + x_3 + x_4 = 0$.

(a) Find a basis for $P$ and explain briefly why it is a basis.

(b) Find a basis for $P^\perp$.

(c) Name a $2 \times 4$ matrix having $P$ as its null space.

4. (5 pts) If $A$ and $B$ are orthogonal matrices, is $AB$ also orthogonal? Why or why not?

5. (10 pts) Use the Gram-Schmidt process to find a set of orthonormal vectors having the same span as $v_1 = (6, 2, 2, 2, 4)$ and $v_2 = (-5, 1, 1, 1, -2)$. Use this to find a least squares solution $v$ to

\[
\begin{pmatrix}
6 & -5 \\
2 & 1 \\
2 & 1 \\
2 & 1 \\
4 & -2
\end{pmatrix}v = \begin{pmatrix}
-1 \\
2 \\
-3 \\
4 \\
2
\end{pmatrix}.
\]

What is the closest point on $\text{Span}(v_1, v_2)$ to $(-1, 2, -3, 4, 2)$?
6. (7 pts) Mark each box Y for yes or N for no. You need not give reasons.

| Are the matrices | \[
\begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{pmatrix}, \\
\begin{pmatrix}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1
\end{pmatrix}, \\
\begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 0
\end{pmatrix}
\] |
<table>
<thead>
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<tbody>
<tr>
<td>invertible?</td>
<td></td>
</tr>
<tr>
<td>orthogonal?</td>
<td></td>
</tr>
<tr>
<td>symmetric?</td>
<td></td>
</tr>
<tr>
<td>skew-symmetric?</td>
<td></td>
</tr>
<tr>
<td>diagonalizable?</td>
<td></td>
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<tr>
<td>stochastic?</td>
<td></td>
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</table>

PART II: DO ANY 4 OUT OF 6 PROBLEMS. (48 pts: 12 each)

7. An \( n \times n \) matrix \( A \) is said to be **idempotent** if \( A^2 = A \).
   (a) Show that if \( A \) is idempotent, then \( I - A \) is too.
   (b) Show that \( v \) is in the column space of \( A \) if and only if \( Av = v \).
   (c) Use (b) to show that the column space of \( A \) equals the null space of \( I - A \). In other words, any vector in the first is also in the second, and vice versa.
   (d) If \( A \) is an \( n \times n \) idempotent matrix of rank \( r \), what is the rank of \( I - A \), and why?

8. (a) Use least squares to find the best straight line \( y = ax + b \) fitting the 4 points \((0,0), (1,8), (3,8), (4,20)\).
   (b) Set up the normal equations for fitting a parabola \( y = ax^2 + bx + c \) to the same points. You don’t need to multiply out or solve them.

9. Suppose \( A \) is a square matrix with real entries.
   (a) If \( A^3 = 0 \), do the eigenvalues of \( A \) all have to be 0? Why or why not?
   (b) If \( A^3 = 0 \) and \( A \) is symmetric, does \( A \) have to be 0? If so, explain why; if not, give a counterexample.
10. Two tanks each contain 100 liters of a salt solution. Initially, the mixture in tank A contains 40 grams of salt while tank B contains 20 grams of salt. Liquid is pumped in and out of the tanks as shown in the figure. Set up a first-order differential equation, and solve it to determine the amount of salt in each tank at time $t$.

11. The weather in Metropolis is always either sunny or cloudy. If it’s sunny in Metropolis today, there is an 80% chance that it will be sunny tomorrow. But if it’s cloudy today, there is only a 60% chance that it will be sunny tomorrow.

(a) Write down the Markov matrix describing this situation. Label the rows and columns appropriately.

(b) If it’s sunny in Metropolis today, what is the probability that it will be sunny in three days later according to the Markov model?

(c) What is a “steady-state vector” for the Markov chain?

(d) Over the long run, what percentage of the time can I expect it to be sunny in Metropolis?

12. Figure 1 represents a top view of a frame made of horizontal 10-foot beams of equal weight. The beams are supported at their ends by 3 pillars A, B, C. You want to place one statue at some point on each of the 3 beams. The statues weigh 130 pounds, 140 pounds, and 180 pounds respectively, but you want to choose the positions so that each pillar bears the same weight. Recall that if a weight $w$ is placed on a horizontal beam, the amount of weight borne at each end is as shown in Figure 2.

(a) Set up a linear system describing the problem, and find the general solution for the positions.

(b) If the distance from statue X to pillar A is 4 feet, what must be the distance from statue Y to pillar B? From statue Z to pillar C?

PART III: DO THE FOLLOWING PROBLEM. (22 pts)

13. (a) Compute a table of the number of walks of length $n$ on the graph below with any given pairs of endpoints. Your answer should be a single $3 \times 3$ symmetric matrix with explicit entries depending on $n$.

As you go along, it’s best to check your work wherever possible—when you invert or diagonalize a matrix, for example. You might also check that the answers you get at the end are integers!

(b) As an example, what is the number of walks of length 10 from A to B?