1. We put the examples in reduced row echelon form:

\[
\begin{pmatrix}
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix} \Rightarrow 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix} \quad \text{So, rank} = 5
\]

\[
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 \\
0 & 0 & 3 & 0 \\
0 & 0 & 4 & 0 \\
0 & 0 & 0 & 5 \\
\end{pmatrix} \Rightarrow 
\begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
\end{pmatrix} \quad \text{So, again, rank} = 5
\]

\[
\begin{pmatrix}
1 & 1 & 1 & 1 \\
2 & 2 & 2 & 2 \\
3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 \\
\end{pmatrix} \Rightarrow 
\begin{pmatrix}
1 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
\end{pmatrix} \quad \text{So, the rank is} \ 1
\]

2. This is true. One way to see this is to note that $A$ and $2A$ have the same reduced row echelon form.

3. a) e.g. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = -A$

b) e.g. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$

c) e.g. $A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

d) Your best guess would be: No.
4. $\vec{v}, \vec{w} \in \mathbb{R}^3$ we non-zero.

a) Suppose $\vec{v} = \vec{w}$ (i.e. $\vec{v} \parallel \vec{w}$). Span $\vec{v}, \vec{w}$ = $3\vec{v} + 4\vec{w}$

   = $3\vec{v} + 4\vec{v}$

   = $7\vec{v}$

   = $\text{Span of } \vec{v}$

i.e. a line $\vec{x} = c\vec{v}$ for $c \in \mathbb{R}$.

b) If $\vec{v} \parallel \vec{w}$ then Span $\vec{v}, \vec{w}$ = $3\vec{v} + 4\vec{w} | c, d \in \mathbb{R}$. This is a plane (we might see this by associating $(c, d) \in \mathbb{R}^2$ to $c\vec{v} + d\vec{w}$ (Span $\vec{v}, \vec{w}$)). We can obtain an equation for points in the plane by geometric considerations. We get: $\vec{x} \cdot (\vec{v} \times \vec{w}) = 0$.

c) We work out (b) in the case $\vec{v} = (1, 2, 3), \vec{w} = (3, 2, 1)$ (i.e. $\vec{v} \parallel \vec{w}$).

$\vec{v} \times \vec{w} = (-4, 8, -4)$

So we have $-4x + 8y - 4z = 0$ or $-x + 2y - z = 0$. 


45 The largest possible rank of a $4 \times 7$ matrix is 4. There are only four rows.

51 Let $A$ be $4 \times 3$. $Ax = b$ cannot be consistent for all $b \in \mathbb{R}^7$. $b$ will be a combination of the three column 4-vectors of $A$, but there will always be vectors in $\mathbb{R}^7$ linearly independent of these column vectors.

53 Underdetermined $\Rightarrow A$ is $m \times n$ with $m < n$. So $m + n - m 
$ therefore always free variables. This means the system either has no solutions or infinitely many.

56 $A$ is $m \times n$, $b_1, b_2 \in \mathbb{R}^m$ s.t. $Ax = b_1$ and $Ax = b_2$ are consistent. Then there are $\bar{x}_1, \bar{x}_2 \in \mathbb{R}^n$ s.t. $A\bar{x}_1 = b_1, A\bar{x}_2 = b_2$.
So $A(\bar{x}_1 + \bar{x}_2) = A\bar{x}_1 + A\bar{x}_2 = b_1 + b_2$ $\Rightarrow A\bar{x} = b_1 + b_2$ is consistent.

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1. a) T  
   b) T  
   c) F  
   d) F  
   e) F

9 \[
\begin{bmatrix}
1 & -1 & 2 \\
0 & 3 & x \\
-1 & 2 & -1
\end{bmatrix} \Rightarrow \begin{bmatrix}
1 & -1 & 2 \\
0 & 3 & x \\
0 & 1 & 1
\end{bmatrix} \Rightarrow x = 3
\]

13 \[
\{(-1), (-2)\} \quad (-2) = -2(-1) \quad \text{so Span}\{(-1), (-2)\} = \text{Span}\{(1)_2\} = \mathbb{R}^2
\]
There is possible inconsistency. So the set does not span $\mathbb{R}^3$.

40 A spanning set for $\mathbb{R}^n$ must contain at least $n$ vectors.

43 $\mathbf{u}_1, \ldots, \mathbf{u}_k \in \mathbb{R}^n$, $c_1, \ldots, c_k \in \mathbb{R} \setminus \{0\}$

\[
\frac{1}{c_i} \cdot c_i \mathbf{u}_i = \mathbf{u}_i \quad \text{so} \quad \mathbf{u}_1, \ldots, \mathbf{u}_k \in \text{Span} \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}
\]

$\mathbf{c}_i \mathbf{u}_i = \mathbf{c}_i \mathbf{u}_i$ of course so $\text{Span} \{c_1 \mathbf{u}_1, \ldots, c_k \mathbf{u}_k\} \subseteq \text{Span} \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$

So the two are equal.

44 $\mathbf{u}_1, \ldots, \mathbf{u}_k \in \mathbb{R}^n$, $c \in \mathbb{R}$

$\mathbf{u}_1, c \mathbf{u}_2 \in \text{Span} \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \Rightarrow \text{Span} \{\mathbf{u}_1, c \mathbf{u}_2, \mathbf{u}_2, \ldots, \mathbf{u}_k\} \subseteq \text{Span} \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$

$\mathbf{u}_1 = (\mathbf{u}_1, c \mathbf{u}_2) - c \mathbf{u}_2 \Rightarrow \text{Span} \{\mathbf{u}_1, \ldots, \mathbf{u}_k\} \subseteq \text{Span} \{\mathbf{u}_1 + c \mathbf{u}_2, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$

So the two are equal.

49 $A = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_m \end{bmatrix}$ where $\mathbf{a}_1, \ldots, \mathbf{a}_m \in \mathbb{R}^n$. $B$ is obtained by a single row operation from $A$.

There are 2 possibilities:

1) $B = \begin{bmatrix} \mathbf{a}_1 \\ \vdots \\ \mathbf{a}_k \end{bmatrix}$ i.e. exchange of rows. This clearly doesn't change the span.
2) \( B = \begin{bmatrix} a_1 \\ \vdots \\ c_a \\ \vdots \\ a_m \end{bmatrix}, \text{ c.d.e R}. \) This does not change the span by (43).

3) \( B = \begin{bmatrix} a_1 \\ \vdots \\ c_b \\ \vdots \\ a_m \end{bmatrix}, \text{ c.d.e R}. \) this does not change the span by (44).

50 \( R, \) the reduced row echelon form of \( A \) is obtained from \( A \) by a finite sequence of elementary row operations. At each stage, the span is preserved as shown in (44). So the span of the rows of \( R \) is the same as the span of the rows of \( A \).