In the text, read section §5.5 and do the Practice Problems.
Then do the following exercises from the text.
From §5.5, do 1, 12, 15, 22, 25, 26, and 43.
Also do the following.

1. Prove that a product of stochastic matrices is stochastic.
2. Prove that a power of a stochastic matrix is stochastic.
3. Consider a Leslie matrix
   \[ A = \begin{pmatrix}
   a_1 & a_2 & a_3 & \cdots & a_{n-1} & a_n \\
   b_1 & 0 & 0 & \cdots & 0 & 0 \\
   0 & b_2 & 0 & \cdots & 0 & 0 \\
   0 & 0 & b_3 & \cdots & 0 & 0 \\
   \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\
   0 & 0 & 0 & \cdots & b_{n-1} & 0 
\end{pmatrix}, \]
   where \( a_i \geq 0 \) and \( 0 < b_i \leq 1 \).
   Prove that
   \[ \det(\lambda I - A) = \lambda^n - a_1\lambda^{n-1} - b_1a_2\lambda^{n-2} - b_1b_2a_3\lambda^{n-3} - \cdots - b_1b_2b_3\cdots b_{n-1}a_n. \]
   Hints: Proceed by induction on \( n \). Expand along the last column, notice that one of the two cofactors is related to a smaller Leslie matrix, and apply the induction hypothesis to it.

4. Suppose a population of wabbits lives for up to 3 years. They do not bear any offspring until the second year of life. A wabbit in its second year of life bears an average of 4 offspring, while a wabbit in its third year of life bears an average of 16 offspring. However, each wabbit has only a 50% chance of surviving from one year to the next.
   (a) Write down the Leslie matrix \( A \) associated with this population.
   (b) Calculate its eigenvalues \( \lambda_1, \lambda_2, \lambda_3 \), where \( \lambda_1 \) is the dominant (i.e. positive real) eigenvalue. Verify that \( |\lambda_2| < \lambda_1 \) and \( |\lambda_3| < \lambda_1 \).
   (c) Find the eigenvector \( v_1 \) associated to \( \lambda_1 \).
   (d) If \( B = \frac{1}{\lambda_1} A \), and \( v_2, v_3 \) are the other eigenvalues of \( A \), evaluate
   \[ \lim_{k \to \infty} B^k(x_1v_1 + x_2v_2 + x_3v_3). \]
   (e) In the long run, what is the overall annual growth rate of the population in percent?
   (f) In the long run, what percent of the population will consist of wabbits less than one year old? Less than two years old?

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5. Two tanks each contain 100 liters of a salt mixture. Initially, the mixture in tank A contains 40 grams of salt while tank B contains 20 grams of salt. Liquid is pumped in and out of the tanks as shown in Figure 1 (on a separate sheet). Determine the amount of salt in each tank at time \( t \) min.

6. Three masses are connected by a series of springs between two fixed points as shown in Figure 2. Assume that the springs all have the same spring constant (i.e. stiffness) \( k \) and let \( x_1(t), x_2(t), x_3(t) \) represent the displacements of the respective masses at time \( t \).

(a) Derive a system of second-order differential equations which describes the motion of the system.

(b) Solve the system if \( m_1 = m_3 = \frac{1}{3}, m_2 = \frac{1}{4}, k = 1, \) and

\[
\begin{align*}
x_1(0) &= x_2(0) = x_3(0) = 1, \\
x_1'(0) &= x_2'(0) = x_3'(0) = 0.
\end{align*}
\]