STRESSES ON BEAMS

1. The diagram at left is a top view of the beams in a ceiling. The beams are securely attached to the side walls, but the T-joints where two beams meet are weak. We want to know how much weight they bear.

2. All beams are 3 meters long and have the same weight $M$.

3. At each T-joint, the base of the T rests on top of the crosspiece.

4. All T-joints are 1 meter from the end of the crosspiece.

5. The 2 joints at the end of each beam bear the weight of the beam, evenly divided between the 2 joints.

6. In addition, they bear the weight borne by the joints resting on that beam.

7. Fact: if a beam of length $y$ is evenly supported at both ends, and a weight $w$ is applied at
distance \( x \) from the left, then \( \frac{x}{y} \) \( w \) of this weight is borne by the right support, and
\( \frac{y-x}{y} \) \( w \) is borne by the left support.

(8) 90° symmetry \( \Rightarrow \) all 4 joints labeled with the same # bear the same weight.

(9) Let \( W_1, W_2, W_3, W_4 \) be the weights borne at the joints labeled 1, 2, 3, 4. These are the 4 unknowns we want to calculate.

(10) Using (5),(6), and (7), we get the linear system

\[
\begin{align*}
W_1 &= \frac{2}{3} W_1 + \frac{1}{3} W_2 + \frac{1}{2} M \\
W_2 &= \frac{1}{3} W_4 + \frac{1}{2} M \\
W_3 &= \frac{1}{3} W_2 + \frac{1}{3} W_1 + \frac{1}{2} M \\
W_4 &= \frac{1}{3} W_3 + \frac{1}{2} M
\end{align*}
\]

Let \( m = \frac{1}{2} M \) and \( x_i = \frac{1}{3} W_i \). Then equivalently

\[
\begin{align*}
-x_1 - x_2 &= m \\
3x_2 - x_4 &= m \\
-x_1 - 2x_2 + 3x_3 &= m \\
-x_3 + 3x_4 &= m
\end{align*}
\]

This can be solved by row operations and back substitution, to get \((x_1, x_2, x_3, x_4) = m (\frac{19}{12}, \frac{7}{12}, \frac{5}{4}, \frac{3}{4})\) and hence \( W_1 = \frac{19}{8} M, W_2 = \frac{7}{8} M, W_3 = \frac{15}{8} M, W_4 = \frac{9}{8} M. \)