## Mathematics V1202y Calculus IV Review Questions

Here are some open-ended questions to help you prepare for the final exam. The exam will cover the entire course. These questions are not guaranteed to be comprehensive: if a topic does not appear here, it still could be on the final.

- **Definitions.** Know the definitions of: moments and center of mass, Jacobian determinant, parametric curves and parametric surfaces, arclength (of parametric curves) and surface area (of parametric surfaces), scalar fields and vector fields, gradient and curl and divergence, rectangular form and polar form of a complex number, complex exponential and logarithm, complex derivative, holomorphic functions, harmonic functions.
- **Parametric curves.** Know how to parametrize the following: circles, ellipses, graphs of functions f(x), parabolas, hyperbolas, helices, line segments (in  $\mathbb{R}^2$  or  $\mathbb{R}^3$ ). More challenging: what about the intersection of a graph z = f(x, y) and an elliptical cylinder  $x^2/a^2 + y^2/b^2 = 1$ ? What about a circle or ellipse that is not centered at the origin? What about a circle or ellipse that lies on a tilted plane in  $\mathbb{R}^3$ ?
- **Parametric surfaces.** Know how to parametrize the following: Spheres, ellipsoids, cylinders (circular and otherwise), graphs of functions f(x, y), paraboloids, surfaces of revolution, hyperboloids, planes.
- Integrals. Know the definitions of integrals: double integral (as a limit of Riemann sums, 15.1/5), iterated integral, triple integral (again a limit of Riemann sums, 15.7/3), line integral with respect to arclength (16.2/3), line integral of a vector field (16.2/13), surface integral with respect to surface area (16.7/3), surface integral of a vector field (16.7/9), complex line integral.
- Leibniz infinitesimals. What are Leibniz infinitesimals? Do they have a rigorous meaning? How are they helpful to the intuition? Can they help you remember any key statements or definitions, e.g. change of variables or the various forms of the line integral?
- Integrands and domains of integration. What is an integrand? What is a domain of integration? For each type of integral, what kind of object is the integrand (scalar field, vector field, complex function)? For each type of integral, what kind of object is the domain of integration (plane region, space region, parametric curve, parametric surface)?
- **Dimensions.** How many dimensions do the following objects have: intervals, plane regions, space regions, parametric curves, parametric surfaces? What is a 0-dimensional region? What would an integral over such a region be?
- Physical interpretations of derivatives. How can you visually interpret the gradient of a scalar field on R<sup>2</sup>? Why is this difficult to imitate on R<sup>3</sup>? In either case, how can you interpret the gradient in terms of the increase of the function? What is the

physical interpretation of divergence? What about curl? What about  $\partial Q/\partial x - \partial P/\partial y$  in  $\mathbb{R}^2$ ? If a fluid in  $\mathbb{R}^3$  is incompressible, what does that imply about its flow? What would be the analogous statement in  $\mathbb{R}^2$ ?

- Physical interpretations of integrals. How can you interpret each of the integrals in physical or visual terms? E.g. as distance along a curve, or the total amount of heat energy in a region, or work done in moving against a force, or the flow of water through a net? How can you give suitable units to the quantities involved (including the Leibniz infinitesimals)?
- **Positivity.** Which of the integrals are guaranteed to be positive by definition? How does this relate to their physical interpretations? Which are the integrals that involve absolute value or square roots in their definitions? Of the line and surface integrals, which ones remain the same when the orientation is reversed, and which ones change sign? Of the surface integrals, which one can be defined for non-orientable surfaces?
- **Symmetry.** When can you argue on symmetry grounds that (a) an integral is twice its value on some smaller domain; (b) an integral is zero? Give examples for *all* of the integrals we've seen, starting from the good old 1-variable integral (except, of course, that the positive integrals won't be zero).
- Integration of 1. For the integrals whose integrands are scalar fields, what do you get in each case if you integrate 1?
- Fundamental theorems. Know your fundamental theorems: the fundamental theorem of line integrals (in 2 and 3 dimensions), Green's theorem, Stokes's theorem, the divergence theorem, the fundamental theorem of complex line integrals. What are the hypotheses? What patterns do you notice? How are they analogous to the Fundamental Theorem of Calculus? In what cases can you use them to conclude that an integral is zero? What are the clues that tell you it will be easier to use a fundamental theorem than to evaluate an integral directly?
- Other theorems. Some more theorems to know: Fubini's theorem, the change of variables theorem, various theorems on independence of path and conservative vector fields,  $\nabla \times \nabla f = 0$ ,  $\nabla \cdot \nabla \times \mathbf{G} = 0$ , De Moivre's theorem, the Cauchy-Riemann equations, Cauchy's Integral Theorem, Cauchy's Integral Formula.
- Counterexamples. What's a surface that's not orientable? What's a plane vector field (P,Q) that has  $\partial Q/\partial x \partial P/\partial y = 0$  yet is not conservative? What's a space vector field that has divergence zero yet is not a curl?
- **Complex functions.** What are some examples of holomorphic functions of a complex variable? Non-holomorphic functions? Multivalued "functions"? When can you evaluate complex line integrals without calculating them directly? When do you know a complex function has an antiderivative?
- Synonyms. What exact synonyms have we seen in the course? Some examples are conservative=a gradient, antiderivative=primitive, holomorphic=complex differentiable=complex analytic...