Mathematics V1202
Calculus IV

Answers to Midterm Exam #1

October 9, 2006
1:10–2:25 pm

1. $\int_0^1 \int_x^1 \cos(y^2)\,dy\,dx = \int_0^1 \int_0^y \cos(y^2)\,dx\,dy = \int_0^1 (x\cos(y^2))_{x=0}^{x=y}\,dy = \int_0^1 y\cos(y^2)\,dy = (\frac{1}{2}\sin(y^2))_0^1 = \frac{1}{2}\sin 1$.

2. This is twice the area below the line $\theta = \pi/4$ and within the circle $r = \sin\theta$, so it is $2\int_0^{\pi/4}\int_0^{\sin\theta} r\,dr\,d\theta = 2\int_0^{\pi/4} \frac{1}{2}\sin^2\theta\,d\theta = \left(\frac{1}{2}u - \frac{1}{4}\sin 2u\right)_0^{\pi/4} = \pi/8 - 1/4$.

3. (a) There are 6 possibilities for the order of arrival, viz. XYZ, XZY, YXZ, YZX, ZXY, ZYX, all equally likely, so the probability of XYZ is 1/6. (b) The probability density is just 1, and the XYZ region is defined by $0 \le x \le y \le z \le 1$, so the integral is $\int_0^1\int_0^z\int_0^y 1\,dx\,dy\,dz = \int_0^1\int_0^z y\,dy\,dz = \int_0^1 (y^2/2)_{y=0}^{y=z} dz = \int_0^1 z^2/2\,dz = 1/6$.

4. See sketch below. Since wedge lies over region E inside ellipse where $x \ge 0$, volume is

$$\iint_E mx\,dA = \int_{-1}^1 \int_0^{3\sqrt{1-y^2}} mx\,dx\,dy = \frac{9m}{2}\int_{-1}^1 (1-y^2)\,dy = 6m.$$

5. Indeed, this is true for any plane region D. The partials of g are twice those of f, so

$$(\tfrac{\partial f}{\partial x})^2 + (\tfrac{\partial f}{\partial y})^2 + 1 \le 4(\tfrac{\partial f}{\partial x})^2 + 4(\tfrac{\partial f}{\partial y})^2 + 1 = (\tfrac{\partial g}{\partial x})^2 + (\tfrac{\partial g}{\partial y})^2 + 1 \le 4(\tfrac{\partial f}{\partial x})^2 + 4(\tfrac{\partial f}{\partial y})^2 + 4.$$

Taking square roots preserves the inequalities, since the square root is an increasing function. So by the comparison property of double integrals,

$$\iint_D \sqrt{(\tfrac{\partial f}{\partial x})^2 + (\tfrac{\partial f}{\partial y})^2 + 1}\,dA \le \iint_D \sqrt{(\tfrac{\partial g}{\partial x})^2 + (\tfrac{\partial g}{\partial y})^2 + 1}\,dA \le \iint_D \sqrt{4(\tfrac{\partial f}{\partial x})^2 + 4(\tfrac{\partial f}{\partial y})^2 + 4}\,dA,$$

and hence $S \le T \le 2S$.

6. This is a spherical box, so

$$\iiint_S (x^2+y^2+z^2)^2\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^2 \rho^4\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$

$$= \left(\int_0^{\pi/2}\sin\phi\,d\phi\right)\left(\int_0^{\pi/2} 1\,d\theta\right)\left(\int_0^2 \rho^6 d\,\rho\right) = 1\cdot\pi/2\cdot 128/7 = 64\pi/7.$$