

**Mathematics V1208y**  
**Honors Mathematics B**  
**Practice Midterm**

1. If  $A \in M_{m,n}(\mathbf{R})$  and  $B \in M_{n,p}(\mathbf{R})$ , prove that  $(AB)^T = B^T A^T$ .
2. (a) If  $A$  is a square matrix with real entries, prove that  $AA^T$  is symmetric.  
(b) Show that the symmetric matrix  $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}$  cannot be expressed as  $AA^T$  for any square  $A$  with real entries.
3. Let  $A$  be a square matrix with complex entries.  
(a) Prove that  $\det A^* = \overline{\det A}$  by reduction by minors.  
(b) If  $\lambda$  is an eigenvalue of  $A$ , show that  $\bar{\lambda}$  is an eigenvalue of  $A^*$ .
4. In  $C[-1, 1]$  with the Euclidean inner product given by

$$\langle f, g \rangle = \int_{-1}^1 f(x)g(x) dx,$$

find an orthonormal basis for the subspace spanned by  $\{1, x, x^2\}$ .  
(Apply the Gram-Schmidt process to this subspace.)

5. Find a general solution to the following inhomogeneous linear system. When you have put the relevant matrix in reduced row-echelon form, mark it as such. Watch out for arithmetic errors.

$$\begin{cases} w + 3x + y + z = 3 \\ 2w + 4x + y + 2z = 8 \\ 3w + 7x + 2y - z = -1 \end{cases}$$

6. Let  $A$  be a square matrix with real entries. If  $\lambda$  is a complex eigenvalue of  $A$ , prove that  $\bar{\lambda}$  is also an eigenvalue of  $A$ .
7. (a) If  $X, Y$  are vectors in a real Euclidean space, prove that  $\langle X, Y \rangle = 0$  if and only if  $\|X - Y\| = \|X + Y\|$ .  
(b) Give a counterexample showing that this is false in a complex Hermitian space.

**Turn over for answers.**

## Answers to Practice Midterm

1. The  $i, k$  entry of  $AB$  is by definition  $\sum_{j=1}^n a_{ij}b_{jk}$ . So the  $i, k$  entry of  $(AB)^T$  is the same with  $i$  and  $k$  interchanged, namely  $\sum_{j=1}^n a_{kj}b_{ji}$ . On the other hand, the  $i, j$  entry of  $B^T$  is  $b_{ji}$ , and the  $j, k$  entry of  $A^T$  is  $a_{kj}$ , so the  $i, k$  entry of  $B^T A^T$  is  $\sum_{j=1}^n b_{ji}a_{kj}$ , the same as before.
2. (a)  $(AA^T)^T = (A^T)^T A^T = AA^T$ , so this is symmetric.  
 (b)  $\det(AA^T) = \det A \det A^T = (\det A)^2 \geq 0$ , but this matrix has determinant  $-8$ .
3. (a)  $A^* = \overline{A^T}$  and  $\det A^T = \det A$ , so it suffices to show  $\det \overline{A} = \overline{\det A}$ . Proof by induction on  $n$ , where  $A$  is  $n \times n$ . Obvious for  $n = 1$ . If true for  $n - 1$ ,  $\det \overline{A} = \sum_i (-1)^i \overline{a_{i1}} \det \overline{A_{i1}} = \sum_i (-1)^i \overline{a_{i1}} \overline{\det A_{i1}} = \overline{\det A}$ , the 2nd equality by induction, the 3rd since conjugation commutes with taking sums and products.  
 (b) If  $\det(A - \lambda I) = 0$ , then by (a),  $\det(A^* - \overline{\lambda} I) = \det(A^* - \overline{\lambda} I^*) = \det(A - \lambda I)^* = \overline{0} = 0$ .
4.  $\|1\|^2 = \langle 1, 1 \rangle = 2$ , so let  $u_1 = 1/\|1\| = 1/\sqrt{2}$ . Then  $\langle u_1, x \rangle = 0$ , so let  $u_2 = x/\|x\| = x/\sqrt{2/3} = \sqrt{3}x/\sqrt{2}$ . Finally,  $\langle u_1, x^2 \rangle = \sqrt{2}/3$ , and  $\langle u_2, x^2 \rangle = 0$ , so let  $u_3$  be  $x^2 - \sqrt{2}/3 u_1$  divided by its own norm, which is  $(3\sqrt{5})/(2\sqrt{2})(x^2 - 1/3)$ .
5. After standard row operations (show your work!), the augmented matrix becomes

$$\left( \begin{array}{cccc|c} 1 & 0 & -\frac{1}{2} & 0 & 3 \\ 0 & 1 & \frac{1}{2} & 0 & -1 \\ 0 & 0 & 0 & 1 & 3 \end{array} \right),$$

where the left-hand block is in RRE form.

The only free variable is the 3rd. Setting it to 0, we find the particular solution  $(3, -1, 0, 3)$ . More generally, if it is  $y$ , we find the general solution:

$$\{(3 + y/2, -1 - y/2, y, 3) \mid y \in \mathbb{R}\}.$$

6. Proof 1: let  $\chi(t) = \det(A - tI)$  be the characteristic polynomial of  $A$ . Then clearly  $\chi$  has real coefficients, that is,  $\chi(t) = c_0 + c_1 t + c_2 t^2 + \dots \pm t^n$  with  $c_i \in \mathbb{R}$ . Now  $\lambda$  is an eigenvalue if and only if  $\chi(\lambda) = 0$ , but then  $\chi(\overline{\lambda}) = c_0 + c_1 \overline{\lambda} + c_2 \overline{\lambda}^2 + \dots \pm \overline{\lambda}^n = \overline{c_0 + c_1 \lambda + c_2 \lambda^2 + \dots \pm \lambda^n} = \overline{0} = 0$ , so  $\overline{\lambda}$  is also an eigenvalue. Proof 2: say  $X = (x_1, \dots, x_n)$  is the eigenvector,  $AX = \lambda X$ . Let  $\overline{X} = (\overline{x}_1, \dots, \overline{x}_n)$ . Then since  $A$  is real,  $A = \overline{A}$  and  $A\overline{X} = \overline{A}\overline{X} = \overline{AX} = \overline{\lambda X} = \overline{\lambda} \overline{X}$ , so  $\overline{X}$  is an eigenvector with eigenvalue  $\overline{\lambda}$ .
7. (a)  $\|X - Y\| = \|X + Y\| \iff \langle X - Y, X - Y \rangle = \langle X + Y, X + Y \rangle \iff \langle X, X \rangle - \langle Y, X \rangle - \langle X, Y \rangle + \langle Y, Y \rangle = \langle X, X \rangle + \langle Y, X \rangle + \langle X, Y \rangle + \langle Y, Y \rangle \iff 0 = 2\langle Y, X \rangle + 2\langle X, Y \rangle = 4\langle X, Y \rangle \iff \langle X, Y \rangle = 0$ .  
 (b) In  $\mathbb{C}^2$  with the standard Hermitian dot product, let  $X = (i, 0)$  and  $Y = (1, 0)$ . Then  $\langle X, Y \rangle = i$  but  $\|X - Y\| = \|X + Y\| = \sqrt{2}$ . (Note: the proof in (a) no longer works since we no longer have  $\langle X, Y \rangle = \langle Y, X \rangle$ .)