Mathematics V1208y
Honors Mathematics B

Answers to Practice Final

1. If $U \subset \mathbb{R}^n$ is open and $F : \mathbb{R}^n \to \mathbb{R}^m$ is a function, then a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ is the *total derivative* of F at $\mathbf{c} \in U$ if

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{F(\mathbf{c}+\mathbf{h}) - F(\mathbf{c}) - T(\mathbf{h})}{\|\mathbf{h}\|} = \mathbf{0}.$$

2. This is simply

$$\iint_T \phi(r(u,v)) \left\| \frac{\partial r}{\partial u}(u,v) \times \frac{\partial r}{\partial v}(u,v) \right\| du\, dv,$$

if it exists.

3. The determinant is the unique function $\det : M_{nn} \to \mathbb{R}$ such that, if $(a_1, \ldots, a_n)$ denotes any matrix with rows $a_1, \ldots, a_n$, and if $c \in \mathbb{R}$,

(i) $\det(a_1, \ldots, ca_i, \ldots, a_n) = c \det(a_1, \ldots, a_n)$;
(ii) $\det(a_1, \ldots, a_i + b_i, \ldots, a_n) = \det(a_1, \ldots, a_i, \ldots, a_n) + \det(a_1, \ldots, b_i, \ldots, a_n)$;
(iii) $\det(a_1, \ldots, a_n) = 0$ if $a_i = a_j$ for any $i \neq j$;
(iv) $\det(e_1, \ldots, e_n) = 1$.

4. False since real spectral theorem asks A to be symmetric: take e.g. $A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

5. True by complex spectral theorem since $AA^* = AA^T = -A^2 = A^T A = A^* A$.

6. True since $F = \nabla g$ where $g(x,y) = e^{xy}$.

7. False since $\mathrm{div} F = 3$, while the divergence of any curl is 0.

8. If all rows are scalar multiples of the first, then the first row spans the row space. Since $a_{11} = 1$, it is nonzero, so it is a basis, hence the rank is 1. Hence the column space is also 1-dimensional. Again since $a_{11} = 1$, the first column is nonzero, so it spans the column space. Hence all columns are scalar multiples of the first one.

9. If $\mathbf{c} \in U \cap V$, then $\mathbf{c} \in U$ and $\mathbf{c} \in V$. By the definition of openness, there exist ϵ_1 and ϵ_2 such that $B_{\epsilon_1}(\mathbf{c}) \subset U$ and $B_{\epsilon_2}(\mathbf{c}) \subset V$. Let $\epsilon = \min(\epsilon_1, \epsilon_2)$. Then $B_\epsilon(\mathbf{c}) \subset B_{\epsilon_1}(\mathbf{c}) \cap B_{\epsilon_2}(\mathbf{c}) \subset U \cap V$.

10. Easy way: note that $f(u,v) = \ln((e^u \cos v)^2 + (e^u \sin v)^2) = 2u$, so partials are 2 and 0 respectively.

Hard way: By the chain rule, $D_f(1,0) = D_g(h(1,0))D_h(1,0)$, that is, if $h = (x,y)$,

$$\begin{pmatrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial g}{\partial x} & \dfrac{\partial g}{\partial y} \end{pmatrix} \begin{pmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{pmatrix},$$

where the partials of the three functions are evaluated at $(u,v) = (1,0)$, $(x,y) = h(1,0) = (e,0)$, and $(u,v) = (1,0)$ respectively. Calculating the partials yields

$$\begin{pmatrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} \end{pmatrix} = \begin{pmatrix} \dfrac{2x}{x^2+y^2} & \dfrac{2y}{x^2+y^2} \end{pmatrix} \begin{pmatrix} e^u \cos v & -e^u \sin v \\ e^u \sin v & e^u \cos v \end{pmatrix},$$

and substituting the values above yields $\partial f/\partial u = 2$, $\partial f/\partial v = 0$.

11. F has $\operatorname{curl} F = \frac{1}{2}\frac{\partial x}{\partial x} + \frac{1}{2}\frac{\partial y}{\partial y} = 1$, so Green's theorem implies $\iint_T 1 = \oint_{\partial T} F \cdot ds$. The line integral is then

$$\int_{-\sqrt{3}}^{\sqrt{3}} F(\gamma(t)) \cdot D_\gamma(t)\, dt = \tfrac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} (t - \tfrac{1}{3}t^3, t^2) \cdot (2t, t^2 - 1)\, dt = \tfrac{1}{2}\int_{-\sqrt{3}}^{\sqrt{3}} (t^2 + \tfrac{1}{3}t^4)\, dt = \frac{8\sqrt{3}}{5}.$$

12. Parametrize the surface by $r : T \longrightarrow S$, $r(u,v) = (u, v, u^2v^2)$, T being the triangle. Then $\partial r/\partial u = (1, 0, 2uv^2)$ and $\partial r/\partial v = (1, 0, 2vu^2)$, so $\partial r/\partial u \times \partial r/\partial v = (-2vu^2, -2vu^2, 1)$. The surface integral is then

$$\iint_T F(r(u,v)) \cdot \left(\frac{\partial r}{\partial u} \times \frac{\partial r}{\partial v}\right) dv\, du = \iint_T (-uv, u^2, u^4v^4) \cdot (-2uv^2, -2vu^2, 1)\, dv\, du$$

$$= \int_0^1 \int_0^{2u} (2u^2v^3 - 2vu^4 + u^4v^4)\, dv\, du = \int_0^1 \left(\frac{2u^2(2u)^4}{4} - \frac{2(2u)^2u^4}{2} + \frac{u^4(2u)^5}{5}\right) du$$

$$= \frac{8}{7} - \frac{4}{7} + \frac{32}{50} = \frac{4}{7} + \frac{16}{25}.$$

13. To prove A linear, observe that if $x = x^S + x^\perp$ and $y = y^S + y^\perp$, then since S and $S^\perp$ are linear subspaces, $x^S + y^S \in S$ and $x^\perp + y^\perp \in S^\perp$. Since $(x^S + y^S) + (x^\perp + y^\perp) = x + y = (x+y)^S + (x+y)^\perp$, by the uniqueness of orthogonal decomposition, $(x^S + y^S) = (x+y)^S$ and $(x^\perp + y^\perp) = (x+y)^\perp$. Hence $A(x+y) = (x+y)^S - (x+y)^\perp = (x^S + y^S) - (x^\perp + y^\perp) = A(x) + A(y)$. Similarly, if $\lambda \in \mathbb{R}$, then $\lambda x^S \in S$ and $\lambda x^\perp \in S^\perp$, so again by uniqueness $\lambda x^S = (\lambda x)^S$ and $\lambda x^\perp = (\lambda x)^\perp$, and hence $A(\lambda x) = (\lambda x)^S - (\lambda x)^\perp = \lambda x^S - \lambda x^\perp = \lambda A(x)$. To prove A orthogonal, note that for any $x, y \in V$, $\langle x, y \rangle = \langle x^S + x^\perp, y^S + y^\perp \rangle = \langle x^S, y^S \rangle + \langle x^\perp, y^\perp \rangle$ since the cross-terms vanish. Hence $\langle Ax, Ay \rangle = \langle x^S - x^\perp, y^S - y^\perp \rangle = \langle x^S, y^S \rangle + \langle x^\perp, y^\perp \rangle = \langle x^S + x^\perp, y^S + y^\perp \rangle = \langle x, y \rangle$, so A is orthogonal by definition.

14. (a) A matrix is orthogonal if and only if its columns are an orthonormal basis. So if the first column is (a, b), it must have length 1, meaning that $a^2 + b^2 = 1$. The second column must be a unit vector in the orthogonal complement of (a, b), a 1-dimensional subspace spanned by $(-b, a)$. If it is $\lambda(-b, a)$, then we must have $1 = \|\lambda(-b, a)\| = |\lambda|(b^2 + a^2) = |\lambda|$, so $\lambda = \pm 1$ leading to the two possibilities stated. (b) Since $a^2 + b^2 = 1$, there exists θ such that $a = \cos\theta$ and $b = \sin\theta$. The first matrix then becomes the usual matrix of rotation by the angle θ. On the other hand, let A be the second matrix. To be a reflection, it suffices to have perpendicular $+1$ and -1 eigenspaces, whose dimensions add to that of the whole space (in this case 2), to play the roles of S and $S^\perp$. The linear systems $(A-I)x = 0$ and $(A+I)x = 0$ have solutions $(b, a-1)$ and $(1-a, b)$ respectively. Since $a^2 + b^2 = 1$, neither of these is $(0, 0)$. Hence they are bases of the ± 1 eigenspaces, so A is reflection in the line $(a-1)x - by = 0$.

15. $g(t) = h(t^3, t^2)$, so by the chain rule, $g'(t) = 3t^2(\partial h/\partial x)(t^3, t^2) + 2t(\partial h/\partial z)(t^3, t^2)$. By the theorem on differentiating under the integral sign, $\partial h/\partial x = \int_0^z \frac{\partial f}{\partial x}(x, y)\, dy$, while by the first fundamental theorem of calculus, $\partial h/\partial z = f(x, z)$. So

$$g'(t) = 3t^2 \int_0^{t^2} \frac{\partial f}{\partial x}(t^3, y)\, dy + 2tf(t^3, t^2).$$

16. By Stokes's theorem, the left-hand side equals $\int\int_S \nabla \times (f\nabla g)$. But $\nabla \times (f\nabla g) = \nabla f \times \nabla g + f(\nabla \times \nabla g) = \nabla f \times \nabla g$ since the curl of a gradient is zero.

17. Method 1: let B be the ball bounded by M and N; then $\operatorname{div}(F - G) = 0$, so by the divergence theorem,

$$\begin{aligned} \iint_M (F-G)\cdot dr^2 + \iint_N (F-G)\cdot dr^2 &= \iint_{\partial B} (F-G)\cdot dr^2 \\ &= \iiint_B 0 \\ &= 0; \end{aligned}$$

now expand out the left-hand side by linearity and rearrange.

Method 2: $\partial M = -\partial N$, that is, the boundary curves are the same but with reversed orientation. Since $F - G$ is divergence-free on any open rectangle containing B, it can be expressed as curl H. Then by Stokes's theorem,

$$\iint_M (F-G)\cdot dr^2 = \oint_{\partial M} H \cdot d\gamma = -\oint_{\partial N} H \cdot d\gamma = \iint_N (F-G)\cdot dr^2;$$

again expand out by linearity and rearrange.

18. The divergence is 2, so by the divergence theorem, $\iint_{\partial Q} F \cdot dr^2 = \iiint_Q 2 = 2 \text{ vol } Q$, which is clearly independent of position.