Mathematics V1208y
Honors Mathematics B

Practice Final Exam

Turn off all electronic devices.

Brief visits to the men's or women's room (just out the door to your left) are OK, but one at a time only, and you must tell me before you go.

Write your name, "Honors Math B, Prof. Thaddeus," and the number of blue books on the cover of each blue book.

Write your name on the attendance sheet when it comes around.

Write all answers and work in your blue books. Do not hand in this sheet.

On each page you use, write the number of the problem *in a circle in the margin.*

You may do more than one problem on a single page. Just put all numbers by their problems.

You may do the problems out of order, but this is discouraged as it can lead to misgrading.

When there is any doubt, state briefly but clearly what statements from Apostol, lecture, or assignments you are using.

In grading the exams, I will emphasize accuracy, brevity, and clarity. Aim for all three.

PART A: Statements. 4 points each. State concisely and precisely, in a complete sentence:

1. The definition of the total derivative.

2. The definition of the integral, with respect to surface area, of a scalar field ϕ on a parametric surface $r : T \to S \subset \mathbb{R}^3$.

3. The four properties uniquely characterizing the determinant.

PART B: True/False. Say which, and give an ultra-brief proof/disproof: just one or two lines. 4 points each.

4. If A has real entries and $A^T = -A$, then $\mathbb{R}^n$ has an orthonormal basis of A-eigenvectors.

5. If A has real entries and $A^T = -A$, then $\mathbb{C}^n$ has an orthonormal basis of A-eigenvectors.

6. If $F(x, y) = (ye^{xy}, xe^{xy})$, then the line integral $\int_C F \cdot ds$ depends only on the endpoints of C.

7. The vector field $F(x, y, z) = (x, y, z)$ is the curl of another vector field.

PART C: Shorter proofs and computations. 7 points each.

8. Let A be a matrix with $a_{11} = 1$. Prove that if all its rows are scalar multiples of the first row, then all its columns are scalar multiples of the first column.

9. If $U \subset \mathbb{R}^n$ and $V \subset \mathbb{R}^n$ are open, prove that the intersection $U \cap V$ is open.

10. Let $g(x, y) = \ln(x^2 + y^2)$, and let $h(u, v) = (e^u \cos v, e^u \sin v)$. If $f = g \circ h$, evaluate $\partial f/\partial u$ and $\partial f/\partial v$ at $(u, v) = (1, 0)$.

11. If $T \subset \mathbb{R}^2$ is of graph type, define its *area* to be $\iint_T 1$. Show that the area equals $\oint_{\partial T} F \cdot ds$ where $F(x, y) = (-\frac{1}{2}y, \frac{1}{2}x)$. Use this to compute the area enclosed by the parametric curve $\gamma : [-\sqrt{3}, \sqrt{3}] \to \mathbb{R}^2$ given by $\gamma(t) = (t^2, \frac{1}{3}t^3 - t)$.

12. Let S be the portion of the surface $z = x^2y^2$ lying over the triangle in the (x, y)-plane with vertices at $(0, 0)$, $(1, 0)$, and $(1, 2)$. Compute the surface integral $\iint_S F \cdot dS$, where $F(x, y, z) = (-xy, x^2, z^2)$.

PART D: Longer proofs. Do any *four* out of six. 10 points each.

13. The orthogonal decomposition theorem states that, given a finite-dimensional Euclidean space V and a subspace $S \subset V$, for all $x \in V$ there exist unique $x^S \in S$ and $x^\perp \in S^\perp$ such that $x = x^S + x^\perp$. Define the *reflection in* S to be the map $A : V \to V$ given by $A(x) = x^S - x^\perp$. Prove that A is linear and orthogonal.

14. Let $\mathbb{R}^2$ have the standard dot product and basis.

(a) Show that the only orthogonal linear maps $\mathbb{R}^2 \to \mathbb{R}^2$ are those whose matrices are $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ or $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$ for $a, b \in \mathbb{R}$ with $a^2 + b^2 = 1$.

(b) Show that in the first case, the map is a rotation, while in the second case, it is a reflection as defined in the previous problem.

15. If $f(x, y)$ is a C^1 scalar field, state and prove a formula for $g'(t)$, where

$$g(t) = \int_0^{t^2} f(t^3, y)\, dy.$$

Hint: express g in terms of $h(x, z) = \int_0^z f(x, y)\, dy$.

16. If f and g are C^2 scalar fields on $\mathbb{R}^3$, and S is a parametric surface, prove that

$$\oint_{\partial S} f \nabla g = \iint_S \nabla f \times \nabla g.$$

17. Let F and G be C^1 vector fields on $\mathbb{R}^3$ with the same divergence, and let M and N be the upper and lower hemispheres of the unit sphere, oriented outward. Show that

$$\iint_M F \cdot dS - \iint_N G \cdot dS = \iint_M G \cdot dS - \iint_N F \cdot dS.$$

18. If $Q \subset \mathbb{R}^3$ is a closed rectangle and $F(x, y, z) = (x^2 + y^2, 2y + 2z, -2xz)$, show that the surface integral $\iint_{\partial Q} F \cdot dS$ depends only on the volume of Q and not on its position.