

4.3 More on vector fields and line integrals

Lecture 19, on April 5.

Recall: a vector field is $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$, or $F : U \rightarrow \mathbb{R}^n$ with $U \subset \mathbb{R}^n$ open.

Definition 247. If $\gamma : [a, b] \rightarrow U$ is a C^1 path and $C = \gamma[a, b]$, the line integral is

$$\int_C F \cdot d\gamma = \int_a^b F(\gamma(t)) \cdot \gamma'(t) dt.$$

Formally, $d\gamma = \frac{d\gamma}{dt} dt = \gamma'(t) dt$.

Example 248. Consider $\gamma : [0, \pi] \rightarrow \mathbb{R}^2$ defined by $\gamma(t) = (\cos t, \sin t)$ (unit semicircle in quadrants I and II). Then $F(x, y) = (0, x)$. So,

$$\begin{aligned} \int_C F \cdot d\alpha &= \int_0^\pi (0, \cos t) \cdot (-\sin t, \cos t) dt \\ &= \int_0^\pi \cos^2 t dt \\ &= \frac{\pi}{2} \end{aligned}$$

Exercise 249. If $f : [a, b] \rightarrow [c, d]$ is a continuous bijection, either $f(a) = c$ and $f(b) = d$ (in which case it is increasing and we call f a forward reparametrization) or $f(a) = d$ and $f(b) = c$ (in which case it is decreasing and we call f a backward reparametrization). Hint: use the intermediate value theorem.

Proposition 250. *The line integral is independent of forward reparametrization and changes sign under backward reparametrization. That is, if $\tilde{\gamma} = \gamma \circ f$, f piecewise C^1 , then*

$$\pm \int_C F \cdot d\gamma = \int_C F \cdot d\tilde{\gamma}, \quad (7)$$

with the + sign if and only if F is forward.

Proof. By the chain rule, $D_{\tilde{\gamma}}(t) = D_{\gamma \circ f}(t) = D_\gamma(f(t))D_f(t) = D_\gamma(f(t))f'(t)$. So, $F(\tilde{\gamma}(t)) \cdot D_{\tilde{\gamma}}(t) = F(\gamma(f(t))) \cdot D_\gamma(f(t))f'(t)$. We're dotting two vector-valued functions of t , times $f'(t)$. Let $g(s) = F(\gamma(s)) \cdot D_\gamma(s)$. Then by change of variables, in (7),

$$RHS = \int_c^d g(s) ds = \int_a^b g(f(t))f'(t) dt = \int_a^b F(\tilde{\gamma}(t)) \cdot D_{\tilde{\gamma}} dt = LHS$$

(if f is forward) while

$$RHS = \int_c^d g(s) ds = \int_a^b g(f(t))f'(t) dt = - \int_a^b g(f(t))f'(t) dt = -LHS$$

if f is backward. □

Convention: always assume that a curve C comes equipped with a parametrization.

Definition 251. Two parametrizations $\gamma, \gamma \circ f$ have the same orientation if f is forward and the opposite orientation if f is backward.

We write $-C$ for C with opposite orientation so we have

$$\int_{-C} F \cdot d\alpha = - \int_C F \cdot d\alpha.$$

Proposition 252 (Linearity).

$$\int_C (aF + bG) \cdot d\gamma = a \int_C F \cdot d\gamma + b \int_C G \cdot d\gamma.$$

Proposition 253 (Concatenation). If $C = \gamma[a, b]$, $C_1 = \gamma[a, c]$, and $C_2 = \gamma[c, b]$, then

$$\int_C F \cdot d\gamma = \int_{C_1} F \cdot d\gamma + \int_{C_2} F \cdot d\gamma.$$

Proof. Exercises. □

Concatenation notation, not used by Apostol: $C = C_1 \# C_2$.

Theorem 254 (Fundamental Theorem of Line Integrals). If $g : U \rightarrow \mathbb{R}$ is a C^1 scalar field, with $U \subset \mathbb{R}^n$ open, and $\gamma : [a, b] \rightarrow U$ is a path, and $X = \gamma(a)$, $Y = \gamma(b)$, then

$$\int_C \nabla g \cdot d\gamma = g(Y) - g(X).$$

Proof.

$$\begin{aligned} \int_C \nabla g \cdot d\gamma &= \int_a^b \nabla g(\gamma(t)) \cdot D_\gamma(t) dt \\ &= \int_a^b \underbrace{D_g(\gamma(t))}_{1 \times n} \underbrace{D_\gamma(t)}_{n \times 1} dt \\ &= \int_a^b D_{g \circ \gamma}(t) dt \\ &= \int_a^b (g \circ \gamma)'(t) dt \\ &= g \circ \gamma(b) - g \circ \gamma(a) \\ &= g(Y) - g(X) \end{aligned}$$

□

Definition 255. If $\nabla g = F$ is continuous, say g is a potential for F , and F is conservative.

Note 256. If C is constant, then $g + C$ is also a potential for F .

Note 257. The motivation for these names comes from physics. If F is a force field (e.g. an electrical field),

$$\int_C F \cdot ds = \text{work moving a particle along the path } C.$$

So $F = \nabla g$ implies that the work is equal to $g(Y) - g(X)$ and

$$-g = \text{potential to do work.}$$

The work is moving the particle against the force.

Example 258. On \mathbb{R}^3 , $g(\vec{x}) = \frac{c}{\|\vec{x}\|}$,

$$\nabla g(\vec{x}) = F(\vec{x}) = \frac{c\vec{x}}{\|\vec{x}\|^3}$$

is a conservative force.

What vector fields are gradients? If we know something is a gradient, we don't have to evaluate line integrals – we can just use the fundamental theorem if the need arises.

Definition 259. A path $\gamma : [a, b] \rightarrow \mathbb{R}^n$ is closed if $\gamma(b) = \gamma(a)$. If $\gamma[a, b] = c$, write

$$\oint_C G \cdot d\gamma.$$

Note 260. On a closed path, obviously the endpoint chosen for C doesn't matter.

Definition 261. An open $U \subset \mathbb{R}^n$ is path-connected if for all $x, y \in U$, there exists a path $\gamma : [a, b] \rightarrow U$ such that $\gamma(a) = x$ and $\gamma(b) = y$.

Note 262. To prove that something is not path-connected, use the intermediate value theorem to show that a path must go through a region that isn't in U .

Example 263. $B_\epsilon(x)$ is path connected: if $y, z \in B_\epsilon(x)$, let C_1 be the path from y to x and C_2 be the path from x to z . Consider $C = C_1 \# C_2$.

Theorem 264. If U is open and path-connected, $F : U \rightarrow \mathbb{R}^n$ and is a continuous vector field, then the following are equivalent:

1. F is conservative (is a gradient)
2. $\int_C F \cdot d\gamma$ depends only on the endpoints of C
3. $\oint_C F \cdot d\gamma = 0$ for all closed C

Furthermore, if so, choose any $X_0 \in U$. Then $F = \nabla g$ where

$$g(X) = \int_{C_X} F \cdot d\gamma$$

for any path C_X in U from X_0 to X . By 2, g is well-defined.

Note 265. Since any path can be selected, pick the one that makes the integral easiest.

Proof. 1 \Rightarrow 2 follows immediately from FTLL.

2 \Rightarrow 3: If $\gamma(a) = \gamma(b) = X$, let $\tilde{\gamma}(t) = X$ for all $t \in [0, 1]$. Then by 2,

$$\oint_C F \cdot d\gamma = \oint_{\tilde{C}} F \cdot d\tilde{\gamma} = 0$$

since $\tilde{\gamma}'(t) = 0$ for all t .

3 \Rightarrow 2: Say C, \tilde{C} are two curves from X to Y . WLOG, parametrizations are $\gamma, \tilde{\gamma} : [0, 1] \rightarrow U$. Then $\delta : [0, 2] \rightarrow U$ defined by

$$\delta(t) = \begin{cases} \gamma(t) & \text{if } t \in [0, 1] \\ \tilde{\gamma}(2-t) & \text{if } t \in [1, 2] \end{cases}$$

Note that you need to check continuity at $t = 1$, but that's easy with $\epsilon - \delta$. Let $D = \delta[0, 2]$. Then by 3,

$$\begin{aligned} 0 &= \int_D F \cdot d\delta \\ &= \int_C F \cdot d\delta + \int_{\tilde{C}} F \cdot d\delta \\ &= \int_C F \cdot d\gamma + \int_{-\tilde{C}} F \cdot d\tilde{\gamma} \\ &= \int_C F \cdot d\gamma - \int_{\tilde{C}} F \cdot d\gamma. \end{aligned}$$

2 \Rightarrow 1: Let g be as stated. Then it suffices to prove that $F = \nabla g$. Since U is open, for all $X \in U$, there exists an $\epsilon > 0$ such that $B_\epsilon(x) \subset U$. Thus, for all $s \in (-\epsilon, \epsilon)$, the line segment from X to $x + se_i$ is contained in U . Let C be any curve in U from X_0 to X , WLOG parametrized by $[-1, 0]$, and let C_s be parametrized by $\gamma : [0, |s|] \rightarrow \mathbb{R}^n$ given by

$$\gamma(t) = \begin{cases} x + te_i & \text{if } s \geq 0 \\ x - te_i & \text{if } s \leq 0. \end{cases}$$

Then $C \# C_s$ is the curve from X_0 to $X + se_i$, where for $t \geq 0$,

$$\gamma'(t) = \begin{cases} e_i & \text{if } s > 0 \\ -e_i & \text{if } s < 0 \end{cases}$$

Thus, $g(x + se_i) = \int_{C \# C_s} F \cdot d\gamma = \int_C F \cdot d\gamma + \int_{C_s} F \cdot d\gamma$. For $s > 0$,

$$\int_{C_s} F \cdot d\gamma = \int_0^s F(x + te_i) \cdot e_i dt$$

but for $s < 0$,

$$\begin{aligned}\int_{C_s} F \cdot d\gamma &= \int_0^{-s} F(x - te_i) \cdot (-e_i) dt \\ &= - \int_0^s F(x + te_i) \cdot (-e_i) dt \\ &= \int_0^s F(x + te_i) \cdot e_i dt.\end{aligned}$$

(Don't substitute $-t$ for t at home, kids.) Hence,

$$\begin{aligned}\frac{\partial g}{\partial x_i}(x) &= \frac{d}{ds}(g(x + se_i))|_{s=0} \\ &= \frac{d}{ds} \left(\int_{C_s} F \cdot d\gamma \right) |_{s=0} \\ &= \frac{d}{ds} \left(\int_0^s F(x + te_i) \cdot e_i dt \right) |_{s=0} \\ &= \frac{d}{ds} \left(\int_0^s F_i(x + te_i) dt \right) |_{s=0} \\ &= F_i(x).\end{aligned}$$

□