Mathematics V1208y
Honors Mathematics B

Answers to Midterm Exam

March 10, 2010

1. A matrix is in reduced row-echelon form if: (i) in each row, the first nonzero entry (if any) is 1, called a *leading 1*; (ii) each leading 1 is the the right of those above it; (iii) each leading 1 is the only nonzero entry in its column.

2. (a) The characteristic polynomial is $(-4-\lambda)(5-\lambda)+18 = \lambda^2+\lambda-2 = (\lambda-2)(\lambda+1)$, so eigenvalues are 2 and -1. The null spaces of $A-2I$ and $A+I$ have bases $(1,1)$ and $(2,1)$ respectively (one can do this by Gauss-Jordan elimination on the respective matrices, or just write them down and eyeball it), so the change of basis from the basis of eigenvectors to the standard basis is $\left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)$. We can invert this using $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \frac{1}{ad-bc}\left(\begin{smallmatrix} d & -b \\ -c & a \end{smallmatrix}\right)$ to express A as $\left(\begin{smallmatrix} -4 & 6 \\ -3 & 5 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 & 0 \\ 0 & -1 \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 2 \\ 1 & -1 \end{smallmatrix}\right)$.

(b) Multiply out $A^k = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2 & 0 \\ 0 & -1 \end{smallmatrix}\right)^k\left(\begin{smallmatrix} -1 & 2 \\ 1 & -1 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 1 & 2 \\ 1 & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} 2^k & 0 \\ 0 & (-1)^k \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & 2 \\ 1 & -1 \end{smallmatrix}\right)$ to obtain the upper right-hand entry $2^{k+1} + 2(-1)^{k+1}$.

3. (a) Elementary row operations on the augmented matrix $(B|I)$ lead to $(I|B^{-1})$ where

$$B^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -6 & 3 \\ 0 & 2 & -1 \end{pmatrix}.$$

(b) Multiplying both sides by B^{-1} shows that the unique solution is $B^{-1}\vec{c} = (3, 4, -1)$.

4. If $n > 2$, then there are at least three rows, so we may subtract row 2 from row 3, then row 2 from row 1 (both elementary row operations of type c, which do not change the determinant) to obtain a matrix with two rows whose entries are all 1. This has zero determinant by axiom (iii) of determinants.

On the other hand, if $n \leq 2$, then $\det(2) = 2$ and $\det\left(\begin{smallmatrix} 2 & 3 \\ 3 & 4 \end{smallmatrix}\right) = -1$.

5. If v is an eigenvector with eigenvalue λ, then $\lambda v = Dv = D^2 v = D\lambda v = \lambda D v = \lambda^2 v$. Since $v \neq 0$, $\lambda = \lambda^2$. Hence $(\lambda-1)\lambda = 0$, so $\lambda = 1$ or $\lambda = 0$.

6. If E has n distinct eigenvalues, then the n associated eigenvectors are independent and hence form a basis. So E is diagonalizable, $E = BDB^{-1}$, where D is diagonal with diagonal entries $\lambda_1, \ldots, \lambda_n$. Then $\det E = \det B \det D \det B^{-1} = \det B\,(\lambda_1 \cdots \lambda_n)/\det B = \lambda_1 \cdots \lambda_n$.

7. The characteristic polynomial is $(1-\lambda)^2 - t = \lambda^2 - 2\lambda + 1 - t$ which, by the quadratic formula, has roots $(2 \pm \sqrt{4-(4-4t)})/2 = 1 \pm \sqrt{t}$. If $t > 0$, these are distinct and real, so there are 2 real independent eigenvectors and A is diagonalizable over $\mathbb{R}$. If $t < 0$, then these equal $1 \pm i\sqrt{-t}$, which are distinct but not real. So there are 2 complex independent eigenvectors, but no real eigenvectors at all, so that A is diagonalizable over $\mathbb{C}$ but not $\mathbb{R}$.