Mathematics V1208y
Honors Mathematics B

Assignment #1
Due January 29, 2010

Reading: Apostol vol. I §16.5–16.19, pp. 583–613.
Or equivalently, Apostol vol. II §2.5–2.19, pp. 36–67.

Don't forget to buy volume II of Apostol!

Of the starred problems, the first four are worth 5 points; the rest are worth 10.

General hint: It will frequently be useful (a) to choose a basis for a subspace and extend to a basis for the whole space; (b) to define a linear map by using the construction principle.

1. Apostol vol. I §16.8 (pp. 589–90) = vol. II §2.8 (pp. 42–44) 4, 6, 10, 22, 23, *24, 25, 27.

2. Apostol vol. I §16.12 (pp. 596–7) = vol. II §2.12 (pp. 50–51) 1, 2ab, *2c, *5, *6, 11, 12, 13.

3. Apostol vol. I §16.16 (pp. 603–4) = vol. II §2.16 (pp. 57–58) 1, 2, 10, *14.

***4.** Let V_n be the vector space of polynomials of degree $\leq n$.

(a) Show that the map $G : V_n \to \mathbb{R}^k$ given by $G(f) = (f(1), f(2), \ldots, f(k))$ is linear, and is surjective when $k \leq n+1$. Hint: consider the polynomials $f(x) = (x-1)(x-2)(x-3)\cdots(x-i+1)(x-i-1)\cdots(x-k)$.

(b) Use rank-nullity to determine the dimension of the subspace of V_n consisting of polynomials where $f(1) = f(2) = \cdots = f(k) = 0$.

5. Show that if two finite-dimensional vector spaces are isomorphic, then they have the same dimension.

6. If V is any linear space and $T : \mathbb{R} \to V$ is linear, show that there exists $u \in V$ such that $Tc = cu$ for all $c \in \mathbb{R}$.

***7.** Let V and W be finite-dimensional vector spaces. Show that a linear map $T : V \to W$ has a right inverse if and only if it is surjective, and a left inverse if and only if it is injective.

***8.** Construct the matrices of the following linear maps with respect to the standard bases:

(a) $A : \mathbb{R}^2 \to \mathbb{R}^2$ is counterclockwise rotation about $(0,0)$ through an angle θ.

(b) $B : \mathbb{R}^3 \to \mathbb{R}^3$ is rotation about the x_3-axis through an angle θ.

(c) $C : \mathbb{R}^2 \to \mathbb{R}^2$ is reflection through the line $x_2 = 2x_1$.

***9.** Suppose V is a finite-dimensional vector space, and $x \in V$, $x \neq 0$. Show there exists a linear $f : V \to \mathbb{R}$ such that $f(x) = 1$. Is f unique?

***10.** Suppose V is a vector space of dimension n, and $W \subset V$ is a hyperplane, that is, a subspace of dimension $n-1$. Show that there is a linear $f : V \to \mathbb{R}$ such that $W = \ker f$. Describe all such f.