Mathematics V1208y
Honors Mathematics B

Answers to Final Examination

May 12, 2010

PART A: True/False. Decide whether the given statement is true or false, and give a brief reason for your answer (sketch of proof or counterexample). 4 points each.

1. True: if $A^T = -A$ and $B = A^{-1}$, then $I = (AB)^T = B^T A^T = -B^T A$, so $B^T = -A^{-1} = -B$.

2. False: try $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

3. True: if $\sum a_i \mathbf{v}_i = 0$, then for each j, $0 = \mathbf{v}_j \cdot \sum a_i \mathbf{v}_i = \sum a_i \, \mathbf{v}_j \cdot \mathbf{v}_i = a_j \|\mathbf{v}_j\|^2$, but $\|\mathbf{v}_j\|^2 \neq 0$, so $a_j = 0$.

4. True: in fact its total derivative at every $\mathbf{v}$ is itself, for certainly

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{L(\mathbf{v}+\mathbf{h}) - L(\mathbf{v}) - L(\mathbf{h})}{\|\mathbf{h}\|} = \lim_{\mathbf{h}\to\mathbf{0}} \frac{\mathbf{0}}{\|\mathbf{h}\|} = \mathbf{0}.$$

5. False: C^1 implies differentiability, which implies continuity.

6. True: the vector field is closed, and the set is star-shaped.

PART B: Shorter proofs and computations. 7 points each.

7. Proof 1: Multiplying the second equation (on the left) by $\mathbf{v}^T$ yields $\mathbf{v}^T\mathbf{w} - \mathbf{v}^T A\mathbf{w} - \mathbf{v}^T A^T \mathbf{w} = 0$. Transposing the first equation and then multiplying (on the right) by $\mathbf{w}$ yields $\mathbf{v}^T\mathbf{w} + \mathbf{v}^T A^T \mathbf{w} + \mathbf{v}^T A\mathbf{w} = 0$. Adding these two equations yields $2\mathbf{v}^T\mathbf{w} = 0$ and hence $\mathbf{v}\cdot\mathbf{w} = 0$.

Proof 2: Let $B = A + A^T$. Then $B^T = A^T + (A^T)^T = A^T + A = B$, so B is symmetric. If $\mathbf{v}$ or $\mathbf{w}$ is $\mathbf{0}$, the statement is trivial. Otherwise, the first equation says that $\mathbf{v}$ is an eigenvector of B with eigenvalue -1, while the second says that $\mathbf{w}$ is an eigenvector of B with eigenvalue 1. But we know that eigenvectors of symmetric matrices having distinct eigenvalues are orthogonal.

8. For any $\mathbf{x} \in S$, take $\epsilon = |f(\mathbf{x})|$ in the definition of continuity; then there exists δ such that $\|\mathbf{y} - \mathbf{x}\| < \delta$ implies $|f(\mathbf{y}) - f(\mathbf{x})| < |f(\mathbf{x})|$, and hence $f(\mathbf{y}) \neq 0$, so that $\mathbf{y} \in S$. Therefore $B_\delta(\mathbf{x}) \subseteq S$.

9. Let $H(s,t) = (s^2 - t^2, s^2 + t^2, st)$; then $g = f \circ H$. By the chain rule, $D_g(s,t) = D_f(H(s,t))\, D_H(s,t)$, or

$$\begin{pmatrix} \dfrac{\partial g}{\partial s} & \dfrac{\partial g}{\partial t} \end{pmatrix} = \begin{pmatrix} \dfrac{\partial f}{\partial x} & \dfrac{\partial f}{\partial y} & \dfrac{\partial f}{\partial z} \end{pmatrix} \begin{pmatrix} 2s & -2t \\ 2s & 2t \\ t & s \end{pmatrix};$$

taking the right-hand entry yields $\partial g/\partial t = -2t\,\partial f/\partial x + 2t\,\partial f/\partial y + s\,\partial f/\partial z$. Here all partial derivatives are to be evaluated at $H(s,t)$.

10. Let $f : U \to \mathbb{R}$ be a C^1 scalar field defined on an open set $U \subset \mathbb{R}^n$, and let $C \subset U$ be a piecewise C^1 curve parametrized by $\gamma : [a,b] \to U$. Then $\int_C \nabla f \cdot d\gamma = f(\gamma(b)) - f(\gamma(a))$.

11. Let $G = (G_1, G_2, G_3)$. Then

$$\begin{aligned} \nabla \cdot (f^2 G) &= \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(f^2 G_i) \\ &= \sum_{i=1}^{3} \Big(\frac{\partial (f^2)}{\partial x_i}\, G_i + f^2\, \frac{\partial G_i}{\partial x_i} \Big) \end{aligned}$$

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\begin{aligned}
&= \sum_{i=1}^{3}\Big(2f\frac{\partial f}{\partial x_i}\,G_i + f^2\,\frac{\partial G_i}{\partial x_i}\Big)\\
&= 2f\,\nabla f\cdot G + f^2\nabla\cdot G.
\end{aligned}
$$

12. The divergence of F is the constant scalar field 2. Let Q be the unit upper ball, $Q = \{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2+z^2 \le 1, z \ge 0\}$, so that the boundary $\partial Q = S \cup -D$, where D is the disc $\{(x,y,z) \in \mathbb{R}^3 \,|\, x^2+y^2 \le 1, z = 0\}$, parametrized in the obvious way by the unit disc in the (x,y)-plane. The disc has outward normal $\mathbf{N} = (0,0,1)$, so on D, $F\cdot\mathbf{N} = 1$ and hence $\iint_D F\cdot d\mathbf{r}^2 = \pi$, the area of the unit disk. Then by the divergence theorem, $\iint_S F\cdot d\mathbf{r}^2 - \iint_D F\cdot d\mathbf{r}^2 = \iiint_Q 2\,dx\,dy\,dz = 4\pi/3$, the last equality simply because a ball of radius r has volume $4\pi/3r^3$. So $\iint_S F\cdot d\mathbf{r}^2 = 4\pi/3+\pi = 7\pi/3$.

PART C: Longer proofs and computations. Do any *four* out of six. 10 points each.

13. Since H is Hermitian, it has real eigenvalues, and by the spectral theorem, H is diagonalizable. So for some A, $AHA^{-1} = D$ where D is diagonal with real entries. But $D - I = A(H-I)A^{-1}$ is also diagonal, and its diagonal entries $d_{ii}-1$ are the eigenvalues of $H-I$. Since $d_{ii}-1$ is imaginary and d_{ii} is real, $d_{ii} = 1$ for all i, so $D = I$, hence $H = I$.

14. By Fubini $k(t) = \int_0^{t^2}\int_0^1 f(x,y)\,dy\,dx$. Let $g(z) = \int_0^z\int_0^1 f(x,y)\,dy\,dx$, and let $h(t) = t^2$. Then $k = g\circ h$. Since $\int_0^1 f(x,y)\,dy$ is a continuous function of x, by the 1st fundamental theorem of calculus g is differentiable and $g'(z) = \int_0^1 f(z,y)\,dy$. Then by the (1-variable!) chain rule, k is differentiable with $k'(t) = 2t\int_0^1 f(t^2,y)\,dy$.

15. Parametrize R by $s : [1,5]\times[-1,1] \to R$ where $s(u,v) = (u-v, u+v)$. Then $\det D_s(u,v) = 2$, so the transformation formula says

$$\iint_R xy\,dx\,dy = \int_{-1}^{1}\int_1^5 2(u^2-v^2)\,du\,dv = 160$$

if I'm not wrong.

16. (a) Straightforward: the answer is $2\mathbf{a}$.

(b) By part (a), using Stokes, this is the line integral $\oint_C H\cdot d\gamma$, where C is the unit circle in the (x,y)-plane. Using your favorite parametrization, say $\gamma(t) = (\cos t, \sin t, 0)$, you find that $H(\gamma(t)) = (0,0,\sin t)$, so $H(\gamma(t))\cdot D_\gamma(t) = 0$, and the line integral is 0.

(c) Moral: If the mouth of your fishnet is in a plane parallel to the motion of the fish, you won't catch any, no matter what shape the net is.

17. The curl of F is $\partial Q/\partial x - \partial P/\partial y = h'(x)+h'(y) \ge 0$. Green's theorem then says that $\oint_{C_r} F\cdot ds = \iint_{D_r}(h'(x)+h'(y))\,dx\,dy$, where D_r is the disc of radius r. For $r \ge 0$, let $f_r(x,y) = h'(x)+h'(y)$ if $x^2+y^2 \le r^2$, 0 otherwise. Then $r' \le r$ implies $f_{r'} \le f_r$, so by comparison

$$
\begin{aligned}
\iint_{D'_r}(h'(x)+h'(y))\,dx\,dy &= \int_{-r}^{r}\int_{-r}^{r} f_{r'}(x,y)\,dx\,dy\\
&\le \int_{-r}^{r}\int_{-r}^{r} f_r(x,y)\,dx\,dy\\
&= \iint_{D_r}(h'(x)+h'(y))\,dx\,dy.
\end{aligned}
$$

18. Since curl $(F-G) =$ curl $F -$ curl $G = 0$, $F-G$ is closed. Then since $\mathbb{R}^3$ is certainly star-shaped, $F-G$ is a gradient, say $\nabla\phi = F-G$. By the fundamental theorem of calculus for line integrals, if $p,q \in \mathbb{R}^3$ are the initial and terminal points of C and D, $\int_C \nabla\phi\cdot ds = \int_D \nabla\phi\cdot dt = \phi(q)-\phi(p)$. Now just substitute $F-G$ for $\nabla\phi$, expand out using linearity, and rearrange.

Note: an alternative approach would be to find a surface bounded by $C\cup D$ and use Stokes's theorem. This is essentially correct, but problematic, for how do you know such a surface exists?