Mathematics V1208y
Honors Mathematics B

Final Examination

May 12, 2010

Turn off all electronic devices.

Brief visits to the men's or women's room (just out the door to your left) are OK, but one at a time only, and you must tell me before you go.

Write your name, "Honors Math B, Prof. Thaddeus," and the number of blue books on the cover of each blue book.

Write your name on the attendance sheet when it comes around.

Write all answers and work in your blue books. Do not hand in this sheet.

On each page you use, write the number of the problem *in a circle in the margin.*

You may do more than one problem on a single page. Just put all numbers by their problems.

You may do the problems out of order, but this is discouraged as it can lead to misgrading.

When there is any doubt, state briefly but clearly what statements from Apostol, lecture, or assignments you are using.

If a region is of graph type; or if a parametrization is counterclockwise; or if a reparametrization is forward or backward, just say so. You don't have to prove it.

In grading the exams, I will emphasize accuracy, brevity, and clarity. Aim for all three.

PART A: True/False. Decide whether the given statement is true or false, and give a brief reason for your answer (sketch of proof or counterexample). 4 points each.

1. If an invertible matrix A is skew-symmetric (i.e. $A_{ji} = -A_{ij}$), then so is its inverse.

2. The eigenvalues of a matrix with rational entries are rational.

3. If the nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ are nonzero and all orthogonal, then they are linearly independent.

4. Any linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable.

5. There exists a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ whose partial derivatives are all continuous, but which is not itself continuous.

6. If $S \subseteq \mathbb{R}^2$ is the union of the two rectangles $(1,2) \times (-1,3)$ and $(-1,3) \times (1,2)$, then the vector field $F(x,y) = (x/(x^2+y^2),\ y/(x^2+y^2))$ is a gradient on S.

PART B: Shorter proofs and computations. 7 points each.

7. Let A be any $n \times n$ matrix with real entries, and suppose two vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ satisfy $\mathbf{v} + A\mathbf{v} + A^T\mathbf{v} = 0$ and $\mathbf{w} - A\mathbf{w} - A^T\mathbf{w} = 0$. Prove that $\mathbf{v} \cdot \mathbf{w} = 0$.

8. Let $f : \mathbb{R}^n \to \mathbb{R}$ be any continuous scalar field. Show that $S = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \neq 0\}$ is an open set.

9. Use the chain rule to express $\frac{\partial g}{\partial t}$ in terms of partial derivatives of $f(x, y, z)$ if $g(s, t) = f(s^2 - t^2, s^2 + t^2, st)$, and $f : \mathbb{R}^3 \to \mathbb{R}$ is a C^1 function.

10. Carefully and completely state the Fundamental Theorem of Line Integrals.

11. Show that if f is a scalar field and G is a vector field on $\mathbb{R}^3$, then $\nabla \cdot (f^2 G) = 2f\, \nabla f \cdot G + f^2\, \nabla \cdot G$. Be sure you understand what each term means!

12. Using the divergence theorem, compute the surface integral $\iint_S F \cdot d\mathbf{r}^2$, where S is the upper hemisphere in $\mathbb{R}^3$ of radius 1, and $F(x, y, z) = (x, y, 1)$. If necessary, you may use the standard formulas for the area of a disc, $A = \pi r^2$, and the volume of a ball, $V = \frac{4}{3}\pi r^3$.

PART C: Longer proofs and computations. Do any *four* out of six. 10 points each.

13. If A is a Hermitian matrix and all the eigenvalues of $A - I$ are imaginary (that is, real multiples of i), prove that $A = I$.

14. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, and let $k(t) = \int_0^1 \int_0^{t^2} f(x, y)\, dx\, dy$. Prove that k is differentiable and express its derivative in terms of a single integral. State clearly what theorems you are using. (Hint: recall we proved that $\int_0^1 f(x, y)\, dy$ is a continuous function of x.)

15. Let R be the rectangle in $\mathbb{R}^2$ with vertices at $(0, 2)$, $(2, 0)$, $(4, 6)$, and $(6, 4)$. Use the transformation formula to express $\iint_R xy\, dx\, dy$ as a double integral with constant limits of integration, and evaluate it.

16. Let $\mathbf{a} = (1, 0, 0) \in \mathbb{R}^3$, and let H be the vector field given by $H(\mathbf{x}) = \mathbf{a} \times \mathbf{x}$.

(a) Compute the curl of H.

(b) If D is the unit disk in the plane, and $\mathbf{r} : D \to S$ is the parametric surface with $\mathbf{r}(u, v) = (u, v, (1 - u^2 - v^2)\, e^{7u^2 v} \cos v)$, compute $\iint_S \mathbf{a} \cdot d\mathbf{r}^2$.

(c) (Optional) Interpret in terms of fish.

17. Let g and $h : \mathbb{R} \to \mathbb{R}$ be increasing C^1 functions, and let F be the vector field on $\mathbb{R}^2$ given by $F(x, y) = (g(x) - h(y),\, h(x) - g(y))$. Use Green's theorem to show that, if C_r is the circle of radius r, then the (counterclockwise) line integral $\oint_{C_r} F \cdot d\gamma$ is an increasing function of r.

18. Let F and G be C^1 vector fields on $\mathbb{R}^3$ having the same curl, and let C and D be C^1 paths in $\mathbb{R}^3$ having the same initial point and the same terminal point. Show that $\int_C F \cdot d\gamma + \int_D G \cdot d\delta = \int_C G \cdot d\gamma + \int_D F \cdot d\delta$.