Power series

Def: For a seq of #s $\{a_n\}$, the seq of fns $\sum_{n=0}^{\infty} f_n(x) = \sum_{n=0}^{\infty} a_n x^n$, where $f_n : \mathbb{R} \to \mathbb{R}$ is $f_n(x) = a_n x^n$, is called the power series centered at 0 with coefficients a_n. Likewise, for $c \in \mathbb{R}$, $\sum_{n=0}^{\infty} a_n (x-c)^n$ is the power series centered at c with coeffs a_n.

Eg $\sum_{n=0}^{\infty} n x^n$, $\sum_{n=0}^{\infty} \left(\frac{x}{3}\right)^n$.

Key question:

For which x does it converge?

Eg we know the geom series

$\sum_{n=0}^{\infty} x^n$ conv $\iff$ $|x| < 1$;

proved conv if $|x| < 1$, div if $|x| = 1$;

for $|x| \geq 1$, easy to show $\lim_{n \to \infty} x^n \neq 0$,
hence $\sum_{n=0}^{\infty} x^n$ div by A9#5 = Prop 3.

Thm: For any power series centered at c, exactly one of the following is true:

(a) it conv for all $x \in \mathbb{R}$;

(b) it conv only for $x = c$;

(c) $\exists R > 0 \mid$
it conv abs when $|x-c| < R$ but it diverges when $|x-c| > R$.

Remarks: May or may not conv for $|x-c| = R$. Call R the <u>radius of convergence</u>. Conventions: $R = \infty$ in case (a), $R = 0$ in case (b).

Lemma: If a power series centered at c conv at x, then it conv absolutely for all $y \in \mathbb{R}$ such that $|y-c| < |x-c|$.

Pf of lemma: Since $\sum a_n(x-c)^n$ conv, by Prop 3, $\lim_{n\to\infty} a_n(x-c)^n = 0$.

Hence $\exists\, N \in \mathbb{N} \mid \forall\, n \in \mathbb{N}$,

$n \geq N \Rightarrow |a_n(x-c)^n| < 1$.

Let $z := \left|\frac{y-c}{x-c}\right|$; then $0 \leq z < 1$, so

$n \geq N \Rightarrow |a_n(y-c)^n| = |a_n(x-c)^n| z^n < z^n$;

hence $\sum_{n=N}^{\infty} |a_n(y-c)^n|$ conv by

comparison with geom series $\sum_{n=N}^{\infty} z^n$;

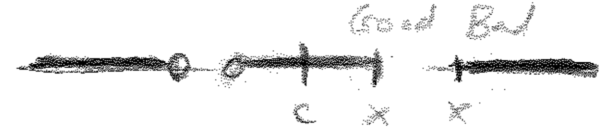
hence $\sum_{n=0}^{\infty} |a_n(y-c)^n|$ conv by

Prop 1 = A9#2. $\boxtimes$

Pf of thm: Clearly no more than one of (a), (b), (c) can be true.

We must show no less than one is true, ie (a), (b) false $\Rightarrow$ (c) true.

Let $S = \{x \in \mathbb{R} \mid \sum_{n=0}^{\infty} a_n(x-c)^n \text{ conv}\}$.



Then:

(i) $c \in S$ since $x = c \Rightarrow x - c = 0$;

(ii) $x \in S$ and $r = x - c \Rightarrow$

$(c-r, c+r) \subset S$,

for $y \in (c-r, c+r) \Rightarrow |y-c| < |x-c|$

$\Rightarrow y \in S$ by the lemma;

(iii) $x \notin S$ and $r = x - c \Rightarrow$

$S \subset [c-r, c+r]$,

for $y \notin [c-r, c+r] \Rightarrow |y-c| > |x-c|$

$\Rightarrow y \notin S$ by (contrapositive of) the lemma.

Now (i) $\Rightarrow S \neq \emptyset$.

And (b) false $\Rightarrow \exists\, x \in \mathbb{R} \mid x \notin S$

$\Rightarrow$ by (iii), S bdd above.

Let $R := \sup S - c$, so that

$c + R = \sup S$.

Note (a) false $\Rightarrow \exists\, x \in S \mid x \neq c$

$\Rightarrow \sup S > c$ by (ii) $\Rightarrow R > 0$.

If $|x-c| > R$, then $\exists\, y \in \mathbb{R} \mid$

$y - c \in (R, |x-c|) \Rightarrow$

$y > c + R = \sup S \Rightarrow$

$y \notin S$. But also

$|x-c| > |y-c|$, so lemma $\Rightarrow x \notin S$.

Hence $S \subset [c-R, c+R]$.

On the other hand, if $|x-c| < R$ then

$c + |x-c| < c + R = \sup S$;

by approx prop, $\exists\, y \in S \mid$

$c + |x-c| < y \leq c + R$

$\Rightarrow |x-c| < y - c \leq R$

$\Rightarrow x \in S$ by the lemma.

Hence $(c-R, c+R) \subset S$. $\boxtimes$

Example: $\sum_{n=0}^{\infty} \frac{n}{2^n} x^n$, centered at 0.

Assume $x > 0$, apply ratio test:

$$\frac{\frac{n+1}{2^{n+1}} x^{n+1}}{\frac{n}{2^n} x^n} = \frac{n+1}{n} \frac{1}{2} x \longrightarrow \frac{x}{2},$$

and $\frac{x}{2} < 1 \iff x < 2$, so $R \geq 2$.

But if $x \geq 2$, $\frac{n}{2^n} x^n \geq n \not\to 0$,

so series div $\Longrightarrow R = 2$.

Thm: Let R be the roc of $\sum_{n=0}^{\infty} a_n (x-c)^n$. If $0 < r < R$, then the series conv *uniformly* on $[c-r, c+r]$.

Pf: Choose $x \in (c+r, c+R)$, so $\sum_{n=0}^{\infty} a_n (x-c)^n$ conv $\Rightarrow \lim_{n\to\infty} a_n (x-c)^n = 0$.

Hence $\exists N \in \mathbb{N} \mid \forall n \in \mathbb{N}$,

$n \geq N \Rightarrow |a_n (x-c)^n| < 1$.

Let $z := \left|\frac{r}{x-c}\right| < 1$.

Then $\forall y \in [c-r, c+r]$,

$n \geq N \Rightarrow$

$$|a_n (y-c)^n| = |a_n (x-c)^n| \left|\frac{y-c}{x-c}\right|^n < z^n.$$

Since $\sum_{n=N}^{\infty} z^n$ is a conv geom

series, $\sum_{n=N}^{\infty} a_n (y-c)^n$ conv unif + abs

for $y \in [c-r, c+r]$ by the

Weierstrass M-test.

Hence same for $\sum_{n=0}^{\infty} a_n (y-c)^n$

by Prop 1. $\boxtimes$

Cor 1: If $\sum_{n=0}^{\infty} a_n(x-c)^n$ has roc $= R$,

then the fn $f: (c-R, c+R)$ thus

defined is conts.

Pf: For any $x \in (c-R, c+R)$,

choose $r < R \mid x \in (c-r, c+r)$.

Then unif conv on $[c-r, c+r] \Rightarrow$

f conts on $[c-r, c+r] \Rightarrow$

f conts at x. ☒

Cor 2: For f as in Cor 1,

$a, b \in (c-R, c+R)$,

$$\int_a^b f(x)\,dx = \sum_{n=0}^{\infty} \int_a^b a_n (x-c)^n\,dx.$$

[Note the RHS integral is easy.]

We say that the power series may

be integrated term by term.

Pf: Unif conv on $[c-r, c+r] \Rightarrow$ thm

from last time on unif conv + integ applies. ☒