Mathematics V1207x
Honors Mathematics A

SOME TIPS ON PROOF-WRITING STYLE

(1) A mathematical proof should be, above all, a convincing argument. It is meant to be read and understood by a human being, not a machine (the automation of theorem-proving is still in its infancy). Therefore, **it should be a coherent exposition written in the English language**, not in computer code or anything resembling it. Of course, it must make use of mathematical terms and symbols. Each symbol, however, is a part of speech: variables like x and y are nouns, while symbols like $=$ and $<$ are verbs. Taking account of this, write in sentences that are complete and grammatically correct. More broadly, keep the reader's perspective in mind. Judge your writing by asking whether a typical reader will find it enlightening and pleasant to read.

(2) Many general rules of good writing apply to the writing of proofs. For example, **sentence structure should be as clear as possible**. The function of each clause, and its relationship to the rest of the sentence, should be apparent. It may help to keep most sentences short, but truly graceful prose contains a variety of sentence structures and lengths.

(3) Likewise, **signal the overall structure of the argument** with a few well-placed sentences. A proof by induction, for example, has two parts: the first dealing with the case $n = 0$ and the second providing the induction step. It helps to divide the proof into two corresponding paragraphs, which might begin as follows.

"Proceed by induction. Suppose first $n = 0$."
"Assume now that the statement is true for some particular n. It suffices to show that it is also true for $n + 1$."

(4) **Avoid wordy or bloated constructions**. Replace unduly long phrases with concise alternatives.

Bad: In order to find the solution of this equation, we can use one of two different methods.

Good: To solve this equation, we can use one of two methods.

Here "in order to" became "to," "find the solution of" became "solve," and "different" was eliminated.

(5) Yet **don't prune a sentence so ruthlessly that it becomes difficult to interpret**.

Very bad: Assume G a group.

Bad: Assume G is a group.

Good: Assume that G is a group.

(6) To be convincing by mathematical standards means that every statement should be justifiable by a previously established fact: a previous theorem, a previous definition, or a rule of logic. Some of us, in high school, may have written "two-column" proofs where every statement in the left column is explicitly justified by a reason in the right column. In principle this should be possible for any proof, but in practice **giving an explicit reason for every step quickly becomes unwieldy and pedantic**. Even some of the steps themselves may be omitted if the reasoning behind them is completely routine. But check with care that it really is routine!

For example, we will soon prove that, for equations involving real numbers, the standard manipulations of high-school algebra, such as multiplying both sides by a constant, are valid. It is therefore legitimate to say the following in a proof.

Good: Solving $\frac{3x}{x^2+2} = 1$, we find $x = 1$ or $x = 2$.

Indeed, this is far preferable to a well-written but long-winded passage such as the following.

Bad: Multiplying both sides by $x^2 + 2$, we find $x^2 + 2 = 3x$. Subtracting $3x$ from both sides, we find $x^2 - 3x + 2 = 0$. Factoring the left-hand side, we find $(x-1)(x-2) = 0$. If the product of two real factors is 0, then one factor must be 0. Hence $x - 1 = 0$ or $x - 2 = 0$. Adding constants to both sides of both equations, we conclude $x = 1$ or $x = 2$.

Likewise, if two passages in a proof are completely parallel (such as those proving the two directions in a biconditional statement), it is acceptable, indeed preferable, to replace the second passage with a remark such as "The reverse direction is proved similarly." Provided, of course, that this is really true!

(7) On the other hand, don't be so concise that the key points of the proof are glossed over. **Give clear and explicit reasons for the important steps in the proof.**

Bad: Since $m = 2i + 1$ and $n = 4j + 2$, we find $m \neq n$.

Good: We know $m = 2i + 1$ is odd and $n = 2(2j + 1)$ is even, so $m \neq n$.

(8) **The passive voice should seldom be used**, and the first-person singular "I" should be avoided entirely. The first-person plural "we" has become standard: think of it as referring to the author and the reader together.

Bad: "Solving the equation, it is found that the roots are real."
[This is passive; moreover, "solving" is a dangling participle, not modifying any word.]

Bad: "Solving the equation, the roots are real."
[Again a dangling participle.]

Bad: "Solving the equation, I find that the roots are real."

Good: "Solving the equation, we find that the roots are real."

Good: "Solving the equation shows that the roots are real."

Good: "Solving the equation yields real roots."

(9) **Use abbreviations in moderation.** To save time, mathematical lectures make liberal use of abbreviations, such as $\forall$, $\exists$, $|$, $\Longrightarrow$, iff, wrt, and wlog. This is also acceptable on examinations, where time is short. But in other mathematical writing, abbreviations should be used sparingly and only when they improve clarity. Quite often, the reader benefits from seeing a phrase written out.

Bad: $\forall\ \epsilon > 0\ \exists\ \delta > 0\ |\ |x - c| < \delta \Longrightarrow |f(x) - f(c)| < \epsilon$.

Good: For all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$.

On the other hand, this can be carried too far.

Bad: Let $\mathbf{N}$ be the set of integers n such that n is at least 0.

Good: Let $\mathbf{N} = \{n \in \mathbf{Z} \,|\, n \geq 0\}$.
[Here symbolic notation is more familiar and clear.]

(10) **"Hence," "therefore," and "consequently" are synonyms** and may be used interchangeably, according to taste, to denote "this implies that...". "So" is also acceptable for this purpose but is more informal.

(11) **If at all possible, distinct formulas or expressions should be separated by text.**

Bad: Consider $f(x), x > 0$.

Good: Consider $f(x)$, where $x > 0$.

(12) **A letter in apposition to a phrase should not be set off by commas.**

Bad: If the discriminant, Δ, is nonnegative, then the roots are real.

Good: If the discriminant Δ is nonnegative, then the roots are real.

(13) **Never begin a sentence with a symbol. It can cause all sorts of confusion.**

Bad: $\cos x$ is an even function.

Good: The function $\cos x$ is even.

Good: We know that $\cos x$ is an even function.

Bad: $x^2 - 1$ has distinct roots.

Good: The polynomial $x^2 - 1$ has distinct roots.

(14) **Beware of trying to pack more than one assertion** (such as a definition and a statement) **into a single equation or clause.** Break it up into two parts.

Very bad: If $\Delta = b^2 - 4ac \geq 0$, then the roots are real.
[A single equation is doing double duty: defining Δ and then placing a condition on it.]

Still fairly bad: If $\Delta = b^2 - 4ac$ is nonnegative, then the roots are real.

Good: Let $\Delta = b^2 - 4ac$. If $\Delta \geq 0$, then the roots are real.

(15) **If a hypothesis begins with "if," then the conclusion should begin with "then."**

Very bad: If $x^2 > 0$, $x > 0$.

Still fairly bad: If x^2 is positive, x is positive.

Good: If $x^2 > 0$, then $x > 0$.

Very bad: If $x = 1$, $y = 2$, $z = 3$.

Good: If $x = 1$ and $y = 2$, then $z = 3$.

(16) **Avoid sentences whose grammar is ambiguous before the end is reached.**

Bad: In the theory of rings, groups and other algebraic structures are treated.

(17) **Mathematical statements, being eternal, should take the present tense.**

Bad: We showed in the previous exercise that X was singular.

Good: We showed in the previous exercise that X is singular.

(18) **Important or elaborate formulas should be displayed on a line by themselves** but should still be part of a complete sentence. For example, no formula is more important than

$$e^{i\pi} + 1 = 0.$$

(19) **If a statement implicitly involves quantifiers** ("for all" or "there exists"), **be sure the meaning is clear.** It is very common to put quantifying phrases at the end instead of the beginning; this is acceptable as long as it causes no doubt about the order of quantifiers.

Bad: We know that $f(x) = 0\,(x \in X)$.
[Is this for all $x \in X$, or just for some $x \in X$?]

Acceptable: We know that $f(x) = 0$ for all $x \in X$.

Good: For all $x \in X$, we know $f(x) = 0$.

Bad: There exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$ for all $\epsilon > 0$.

Good: For all $\epsilon > 0$, there exists $\delta > 0$ such that $|x - c| < \delta$ implies $|f(x) - f(c)| < \epsilon$.

(20) **Don't use the same notation for two different things.**

Bad: Let $a_i = e^{2\pi i/n} b_i$.
[Here i is both an index and $\sqrt{-1}$.]

Good: Let $a_j = e^{2\pi i/n} b_j$.

(21) **Don't use different notations for the same thing without a good reason.**

Bad: Choose x_i for $1 \leq i \leq n$ and consider $\sum_{j=1}^{n} x_j$.

Good: Choose x_i for $1 \leq i \leq n$ and consider $\sum_{i=1}^{n} x_i$.

A good reason might be a clever substitution or change of variables that advances the argument.

(22) **Don't let indices, subscripts, or superscripts become needlessly elaborate.** Even if some quantity a depends on a variable x and an index i, it might be more clear to denote it simply by a rather than by $a_i(x)$, depending on circumstances. When notation becomes too heavy, consider how it might be trimmed down while keeping the argument rigorous.

Many of these tips, and many of the examples illustrating them, are paraphrased from other sources, which are gratefully acknowledged below.

Some Hints on Mathematical Style by David Goss,
`https://people.math.osu.edu/goss.3/hint.pdf`

Mathematical Writing by Donald E. Knuth, Tracy Larrabee, and Paul M. Roberts,
`http://tex.loria.fr/typographie/mathwriting.pdf`

Writing a Math Phase Two Paper by Steven L. Kleiman,
`http://www.mit.edu/afs/athena.mit.edu/course/other/mathp2/www/piil.html`