Mathematics V1207x
Honors Mathematics A

Answers to Practice Midterm

October 28, 2015

1. For $S \subset \mathbb{R}$ nonempty and bounded above, let $s = \sup S$; then for all $\epsilon > 0$, there exists $x \in S$ such that $x > s - \epsilon$.

2. For any s and $s' \in S$, suppose that $g \circ f(s) = g \circ f(s')$. Then $g(f(s)) = g(f(s'))$ by definition of composition. Since g is injective, $f(s) = f(s')$. Then since f is injective, $s = s'$. Hence $g \circ f$ is injective.

3. Proof by induction on n. For $n = 0$, clearly $2^{0+1} = 2 \geq 2 = 2\dot{0} + 2$. Now assume that for a given n, $2^{n+1} \geq 2n + 2$. Then $2^{n+2} = 2 \cdot 2^{n+1} = 2^{n+1} + 2^{n+1} \geq 2n + 2 + 2n + 2 = 2(n+1) + 2 + n \geq 2(n+1) + 2$ where the last inequality is because $n \geq 0$ for all natural n.

4. (a) $f(x) = [x]$; (b) $f(x) = x$. (c): If $x \leq y$, then $S \cap [0, x] \subset S \cap [0, y]$, so any upper bound (such as $\sup S \cap [0, y]$) for the latter is an upper bound for the former, so the least upper bound must satisfy $\sup S \cap [0, x] \leq \sup S \cap [0, y]$, so $f(x) \leq f(y)$. Hence f is monotonic, hence integrable on $[0, 10]$.

5. Proof 1: For all $x \in \mathbb{R}$, $|g(x)| \leq 1$, so $-1 \leq g(x) \leq 1$, so by the comparison theorem for integrals, $-(b - a) = \int_a^b -1\, dx \leq \int_a^b g(x)\, dx \leq \int_a^b 1\, dx = b - a$, so $|\int_a^b g(x)\, dx| \leq b - a$.
Proof 2: By an assigned problem (A5#6 in fact) g integrable implies $|g|$ integrable, and by another assigned problem (A4#5) and the comparison theorem for integrals, $|\int_a^b g(x)\, dx| \leq \int_a^b |g(x)|\, dx \leq \int_a^b 1\, dx = b - a$.
(Proof 1 is perhaps better than Proof 2 since it doesn't use g integrable implies $|g|$ integrable, which is relatively hard to prove.)

6. Proof 1: If there were a limit, say $K \in \mathbb{R}$, then by the definition of limit, for all $\epsilon > 0$ there exists $\delta > 0$ such that $0 < |x| < \delta$ implies $|1/x - K| < \epsilon$, that is, $-\epsilon < 1/x - K < \epsilon$. For any such ϵ and δ, by the Archimedean property there exists an integer $n > \max(1/\delta, \epsilon + K)$, that is, $n > 1/\delta$ and $n > \epsilon + K$. Then $x = 1/n$ satisfies $0 < |x| < \delta$ but $1/x - K > \epsilon$, contradiction.
Proof 2: If there were a limit, say $K \in \mathbb{R}$, then basic limit theorem 1c implies $1 = \lim_{x \to 0} x \frac{1}{x} = (\lim_{x \to 0} x)(\lim_{x \to 0} \frac{1}{x}) = 0K = 0$, contradiction.

7. Proof 1: since all terms in the inequalities are nonnegative, $|fg(x)| = |f(x)|\,|g(x)| \leq |x| \cdot 1 = |x|$, so $-|x| \leq fg(x) \leq |x|$. Since $\lim_{x \to 0} -|x| = \lim_{x \to 0} |x| = 0$, by the squeezing theorem $\lim_{x \to 0} fg(x) = 0$. But $|f(0)| \leq |0|$, so $f(0) = 0$ and $fg(0) = 0g(0) = 0$. Hence $\lim_{x \to 0} fg(x) = fg(0)$.
Proof 2: Given $\epsilon > 0$, take $\delta = \epsilon$. Then $0 < |x - 0| < \delta$ implies $|fg(x)| = |f(x)|\,|g(x)| \leq |x| \cdot 1 < \delta = \epsilon$, so $\lim_{x \to 0} fg(x) = 0$. But $|f(0)| \leq |0|$, so $f(0) = 0$ and $fg(0) = 0g(0) = 0$. Hence $\lim_{x \to 0} fg(x) = fg(0)$.