1. True: this is $g \circ f(x)$ where $g(y) = |y|$ is also continuous.

2. False: not even true for $f(x) = x$ on $[-1, 1]$.

3. False: the harmonic series $a_n = 1/n$ is a counterexample.

4. True: $e^{-(n+1)^2}/e^{-n^2} = e^{-2n-1} \to 0$ as $n \to \infty$, so converges by ratio test.

5. True: $|\sin(nx)/2^n| \leq 1/2^n$, and $\sum 1/2^n$ converges, so this converges uniformly by Weierstrass M-test. And uniform limits of continuous functions are continuous.

6. True: we proved in class that the map taking $A$ to $T_A$ is an isomorphism, hence linear and injective, hence has kernel = \{0\} by A12#4.

7. If $f : [a, b] \to \mathbb{R}$ is continuous, then for all $\epsilon > 0$ there exists a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that for each $i \in \{1, \ldots, n\}$, $M(f|_{x_{i-1}, x_i}) - m(f|_{x_{i-1}, x_i}) < \epsilon$. Here $M$ and $m$ denote the absolute maximum and absolute minimum, respectively.

8. If $f_n : [a, b] \to \mathbb{R}$ are continuous and converge uniformly to $f : [a, b] \to \mathbb{R}$, then the indefinite integrals $\int_a^x f_n(t) \, dt$ converge uniformly, as a function of $x$, to $\int_a^x f(t) \, dt$.

9. A series $\sum_{n=0}^{\infty} a_n$ converges if the sequence $\{\sum_{n=0}^m a_n\}$ of partial sums converges, that is, if $\lim_{m \to \infty} \sum_{n=0}^m a_n$ exists. It converges absolutely if the series $\sum_{n=0}^{\infty} |a_n|$ converges.

10. The standard basis vectors $\vec{e}_1, \ldots, \vec{e}_n \in \mathbb{R}^n$ are the vectors whose components are $(\vec{e}_i)_j = \delta_{i,j}$. Here, for $i, j \in \{1, \ldots, n\}$, we define $\delta_{i,j} = 1$ if $i = j$ and 0 if $i \neq j$.

11. Since $f$ is a bijection, it has an inverse, that is, a function $h : T \to S$ such that $f \circ h = \text{id}_T$ and $h \circ f = \text{id}_S$. Then $f \circ g = (f \circ g) \circ \text{id}_T = (f \circ g) \circ (f \circ h) = f \circ (g \circ f) \circ h = f \circ (\text{id}_S) \circ h = f \circ h = \text{id}_T$.

12. Differentiable implies continuous, so by the intermediate value theorem, there exist $a \in (0, 1)$ and $b \in (1, 2)$ such that $f(a) = f(b) = 0$. Then by Rolle’s theorem (or the mean-value theorem) there exists $x \in (a, b)$ such that $f’(x) = 0$.

13. Since $|\sin x| \leq 1$, $f(x/2 - 1) < x < f(x/2 + 1)$. Since $f$ is continuous, by the intermediate value theorem there exist $y \in (x/2 - 1, x/2 + 1)$ such that $f(y) = x$.

If there were two distinct $y_1 < y_2$ such that $f(y_1) = f(y_2)$, since $f$ is differentiable there would exist $c \in (y_1, y_2)$ such that $f’(c) = 0$ by Rolle’s theorem. But $f’(x) = \cos x + 2$ is never zero, contradiction.

14. The hypothesis means that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - L| < \epsilon$. By the triangle inequality,

$$|L| = |L - a_n + a_n| \leq |L - a_n| + |a_n|$$
and

\[ |a_n| = |a_n - L + L| \leq |a_n - L| + |L|. \]

Subtracting \( |L - a_n| + |L| \) from the first inequality and \( |L| \) from the second gives

\[-|a_n - L| \leq |a_n| - |L| \leq |a_n - L|, \]

that is,

\[ ||a_n| - |L|| \leq |a_n - L|. \]

Hence for \( \epsilon, N, \) and \( n \) as above, \( ||a_n| - |L|| \leq |a_n - L| < \epsilon \), which implies the conclusion.

Alternative: By A9#8, a continuous function such as \( f(x) = |x| \) takes convergent sequences to convergent sequences; more precisely, if \( \lim_{n \to \infty} a_n = L \), then \( \lim_{n \to \infty} f(a_n) = f(L) \), hence \( \lim_{n \to \infty} |a_n| = |L| \).

15. Pointwise: Take \( x \in [0, 1] \). If \( x = 0 \), then \( f_n(x) = 0 \) for all \( n \), so obviously \( f_n(x) \to 0 \). Otherwise, for any \( \epsilon > 0 \), let \( N \) be an integer \( > 1/x \); then \( n \geq N \) implies \( 1/n \leq 1/N < x \), so \( f_n(x) = 0 \) and \( |f_n(x) - 0| = |0 - 0| < \epsilon \).

Uniform: Take \( \epsilon = 1 \); then for all \( n \) there exists \( x \in [0, 1] \) (say \( x = 1/2n \)) such that \( |f_n(x) - 0| = n \geq \epsilon \), so convergence isn’t uniform.

Alternative: \( f_n \) is a step function, and \( \int_0^1 f_n(x) \, dx = n/n = 1 \) doesn’t converge to \( \int_0^1 0 \, dx = 0 \), contradicting A10#3.

16. Let \( E_i \) be the \( i \)th standard basis vector and let \( B \) be the vector with \( i \)th component \( b_i = f(E_i) \). Then for any \( X = \sum_{i=1}^n x_i E_i \), we have \( f(X) = f(\sum_{i=1}^n x_i E_i) = \sum_{i=1}^n x_i f(E_i) = \sum_{i=1}^n x_i b_i = \sum_{i=1}^n b_i x_i = B \cdot X \).

17. (a) We know that \( F([0, 1], \mathbb{R}) \) is a vector space, so it suffices to show that the set of step function is a subspace. But we know that if \( s \) and \( t \) are step functions, then so is \( s + t \) (by passing to a common refinement). And if \( s \) is a step function and \( c \in \mathbb{R} \), then \( cs \) is a step function (by using the same refinement as for \( s \)). Hence the set of step functions is closed under addition and scalar multiplication, so it is a subspace.

(b) This is an immediate consequence of linearity (and homogeneity) for integrals of step functions: \( \int_0^x (s + t)(y) \, dy = \int_0^x s(y) \, dy + \int_0^x t(y) \, dy \) and \( \int_0^x cs(y) \, dy = c \int_0^x s(y) \, dy \). So if we let \( \phi(s) = \int_0^x s(y) \, dy \), then \( \phi(s + t) = \phi(s) + \phi(t) \) and \( \phi(cs) = c\phi(s) \).

(c) This is not injective: both \( s(x) = 0 \) and

\[
t(x) = \begin{cases} 
1 & \text{if } x = 0 \\
0 & \text{if } x \neq 0
\end{cases}
\]

have \( \phi(s) = \phi(t) = 0 \).