Mathematics V1207x
Honors Mathematics A

Answers to Practice Final
December 21, 2015

1. True: this is $g \circ f(x)$ where $g(y) = |y|$ is also continuous.

2. False: not even true for $f(x) = x$ on $[-1, 1]$.

3. False: the harmonic series $a_n = 1/n$ is a counterexample.

4. True: $e^{-(n+1)^2}/e^{-n^2} = e^{-2n-1} \to 0$ as $n \to \infty$, so converges by ratio test.

5. True: $|\sin(nx)/2^n| \leq 1/2^n$, and $\sum 1/2^n$ converges, so this converges uniformly by Weierstrass M-test. And uniform limits of continuous functions are continuous.

6. True: we proved in class that the map taking A to T_A is an isomorphism, hence linear and injective, hence has kernel $= \{\vec{0}\}$ by A12#4.

7. If $f : [a, b] \to \mathbb{R}$ is continuous, then for all $\epsilon > 0$ there exists a partition $a = x_0 < x_1 < \cdots < x_n = b$ such that for each $i \in \{1, \ldots, n\}$, $M(f|_{[x_{i-1},x_i]}) - m(f|_{[x_{i-1},x_i]}) < \epsilon$. Here M and m denote the absolute maximum and absolute minimum, respectively.

8. If $f_n : [a, b] \to \mathbb{R}$ are continuous and converge uniformly to $f : [a, b] \to \mathbb{R}$, then the indefinite integrals $\int_a^x f_n(t)\,dt$ converge uniformly, as a function of x, to $\int_a^x f(t)\,dt$.

9. A series $\sum_{n=0}^{\infty} a_n$ *converges* if the sequence $\{\sum_{n=0}^{m} a_n\}$ of partial sums converges, that is, if $\lim_{m\to\infty} \sum_{n=0}^{m} a_n$ exists. It *converges absolutely* if the series $\sum_{n=0}^{\infty} |a_n|$ converges.

10. The *standard basis vectors* $\vec{e}_1, \ldots, \vec{e}_n \in \mathbb{R}^n$ are the vectors whose components are $(\vec{e}_i)_j = \delta_{i,j}$. Here, for $i, j \in \{1, \ldots, n\}$, we define $\delta_{i,j} = 1$ if $i = j$ and 0 if $i \neq j$.

11. Since f is a bijection, it has an inverse, that is, a function $h : T \to S$ such that $f \circ h = \mathrm{id}_T$ and $h \circ f = \mathrm{id}_S$. Then $f \circ g = (f \circ g) \circ \mathrm{id}_T = (f \circ g) \circ (f \circ h) = f \circ (g \circ f) \circ h = f \circ (\mathrm{id}_S) \circ h = f \circ h = \mathrm{id}_T$.

12. Differentiable implies continuous, so by the intermediate value theorem, there exist $a \in (0, 1)$ and $b \in (1, 2)$ such that $f(a) = f(b) = 0$. Then by Rolle's theorem (or the mean-value theorem) there exists $x \in (a, b)$ such that $f'(x) = 0$.

13. Since $|\sin x| \leq 1$, $f(x/2 - 1) < x < f(x/2 + 1)$. Since f is continuous, by the intermediate value theorem there exists $y \in (x/2 - 1, x/2 + 1)$ such that $f(y) = x$.

If there were two distinct $y_1 < y_2$ such that $f(y_1) = f(y_2)$, since f is differentiable there would exist $c \in (y_1, y_2)$ such that $f'(c) = 0$ by Rolle's theorem. But $f'(x) = \cos x + 2$ is never zero, contradiction.

14. The hypothesis means that for all $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $n \geq N$ implies $|a_n - L| < \epsilon$. By the triangle inequality,

$$|L| = |L - a_n + a_n| \leq |L - a_n| + |a_n|$$

and

$$|a_n| = |a_n - L + L| \le |a_n - L| + |L|.$$

Subtracting $|L - a_n| + |L|$ from the first inequality and $|L|$ from the second gives

$$-|a_n - L| \le |a_n| - |L| \le |a_n - L|,$$

that is,

$$||a_n| - |L|| \le |a_n - L|.$$

Hence for ϵ, N, and n as above, $||a_n| - |L|| \le |a_n - L| < \epsilon$, which implies the conclusion.

Alternative: By A9#8, a continuous function such as $f(x) = |x|$ takes convergent sequences to convergent sequences; more precisely, if $\lim_{n\to\infty} a_n = L$, then $\lim_{n\to\infty} f(a_n) = f(L)$, hence $\lim_{n\to\infty} |a_n| = |L|$.

15. Pointwise: Take $x \in [0, 1]$. If $x = 0$, then $f_n(x) = 0$ for all n, so obviously $f_n(x) \to 0$. Otherwise, for any $\epsilon > 0$, let N be an integer $> 1/x$; then $n \ge N$ implies $1/n \le 1/N < x$, so $f_n(x) = 0$ and $|f_n(x) - 0| = |0 - 0| < \epsilon$.

Uniform: Take $\epsilon = 1$; then for all n there exists $x \in [0, 1]$ (say $x = 1/2n$) such that $|f_n(x) - 0| = n \ge \epsilon$, so convergence isn't uniform.

Alternative: f_n is a step function, and $\int_0^1 f_n(x)\,dx = n/n = 1$ doesn't converge to $\int_0^1 0\,dx = 0$, contradicting A10#3.

16. Let E_i be the ith standard basis vector and let B be the vector with ith component $b_i = f(E_i)$. Then for any $X = \sum_{i=1}^n x_i E_i$, we have $f(X) = f(\sum_{i=1}^n x_i E_i) = \sum_{i=1}^n x_i f(E_i) = \sum_{i=1}^n x_i b_i = \sum_{i=1}^n b_i x_i = B \cdot X$.

17. (a) We know that $\mathcal{F}([0, 1], \mathbb{R})$ is a vector space, so it suffices to show that the set of step function is a subspace. But we know that if s and t are step functions, then so is $s + t$ (by passing to a common refinement). And if s is a step function and $c \in \mathbb{R}$, then cs is a step function (by using the same refinement as for s). Hence the set of step functions is closed under addition and scalar multiplication, so it is a subspace.

(b) This is an immediate consequence of linearity (and homogeneity) for integrals of step functions: $\int_0^x (s + t)(y)\,dy = \int_0^x s(y)\,dy + \int_0^x t(y)\,dy$ and $\int_0^x cs(y)\,dy = c\int_0^x s(y)\,dy$. So if we let $\phi(s) = \int_0^x s(y)\,dy$, then $\phi(s + t) = \phi(s) + \phi(t)$ and $\phi(cs) = c\phi(s)$.

(c) This is not injective: both $s(x) = 0$ and

$$t(x) = \begin{cases} 1 \text{ if } x = 0 \\ 0 \text{ if } x \neq 0 \end{cases}$$

have $\phi(s) = \phi(t) = 0$.