Mathematics V1207x
Honors Mathematics A

Answers to Midterm Exam
October 28, 2015

1. (a) Let $f : [a,b] \to \mathbb{R}$ be bounded. Its *lower integral* is $\underline{I}(f) = \sup \underline{S}(f)$ where

$$\underline{S}(f) = \left\{ \int_a^b s(x)\,dx \mid s : [a,b] \to \mathbb{R} \text{ a step function, } s \le f \right\}.$$

Its *upper integral* is $\overline{I}(f) = \inf \overline{S}(f)$ where

$$\overline{S}(f) = \left\{ \int_a^b t(x)\,dx \mid t : [a,b] \to \mathbb{R} \text{ a step function, } t \ge f \right\}.$$

(b) A function f as in (a) is *integrable* if $\underline{I}(f) = \overline{I}(f)$.

2. Given $\epsilon > 0$, we know there exist $\delta_1, \delta_2 > 0$ such that for all $x \in \mathbb{R}$,

$$\begin{aligned} 0 < |x-c| < \delta_1 &\implies |f(x)-k| < \epsilon \quad \text{and} \\ 0 < |x-c| < \delta_2 &\implies |g(x)-k| < \epsilon. \end{aligned}$$

Take $\delta = \min(\delta_1, \delta_2)$. Then for all $x \in \mathbb{R}$,

$$\begin{aligned} 0 < |x-c| < \delta &\implies 0 < |x-c| < \delta_1 \text{ and } 0 < |x-c| < \delta_2 \\ &\implies |f(x)-k| < \epsilon \text{ and } |g(x)-k| < \epsilon \\ &\implies |h(x)-k| < \epsilon, \end{aligned}$$

regardless of whether or not $x \in \mathbb{Q}$.

3. Let $h(x) = 1 - x$. Being a polynomial, h is continuous.

Suppose f is continuous. Then $g = f \circ h$, as a composition of two continuous functions, is continuous.

Conversely, suppose g is continuous. Substituting $1 - y$ for x, we find $g(1-y) = f(1-(1-y)) = f(y)$ and hence $f = g \circ h$. Then f, as a composition of two continuous functions, is continuous.

4. Let $g(x) = \int_a^x f(t)\,dt$ be the indefinite integral. If $x, y \in [a,b]$, then $g(y) - g(x) = \int_a^y f(t)\,dt - \int_a^x f(t)\,dt = \int_x^y f(t)\,dt$ by concatenation. (The integrals are all defined by A5 #5.) If $x \le y$, by comparison $0 = \int_x^y 0\,dt \le \int_x^y f(t)\,dt = g(y) - g(x)$ and hence $g(x) \le g(y)$.

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5. If $d < c$, then there exists an element of S greater than d (namely c), and hence d cannot be an upper bound for S. Since c is an upper bound, it is therefore the least upper bound.

6. (a) For $n \in \mathbb{N}$, let $[n] = \{i \in \mathbb{N} \mid 0 < i \leq n\}$. A set S is *finite* if there exist $n \in \mathbb{N}$ and a bijection $f : [n] \to S$.

(b) A set is *infinite* if it is not finite.

7. Let $n = \#S$. Since S is finite and nonempty, $n \in \mathbb{N} \setminus \{0\}$. Proof by induction on n, starting with $n = 1$. (To conform to the convention in class that induction starts with 0, one could proceed by induction on $n - 1$.)

If $n = 1$, then there is a bijection $f : [1] \to S$, so $S = \{f(1)\}$. Then $f(1)$ is clearly an upper bound for S, and no $d < f(1)$ can be an upper bound, so $\sup S = f(1) \in S$.

Now, for a given $n > 0$, assume the statement for sets with n elements and suppose there is a bijection $f : [n+1] \to S$. Let $T = \{f(i) \mid 0 < i \leq n\} = \{f(1), \ldots, f(n)\}$. By induction $\sup T \in T$. Let $c = \max(\sup T, f(n+1))$. Then c is an upper bound and is an element of S, since it equals either $\sup T$ or $f(n+1)$. By #**5**, $c = \sup S$.

Note: it is tempting to arrange the elements of S in increasing order as $x_1 < x_2 < \cdots < x_n$ and then prove that $x_n = \sup S$. However, it is hard to prove that the elements can be arranged in increasing order without using the result of this problem! Thus there is a grave risk of circularity.