A SPEEDY REVIEW OF LOGIC

Letters $P$, $Q$, $R$ denote statements which can be either true or false, but not both. For example, “All math teachers have red hair,” “Crunchy-Wunchies are fortified with vitamins,” “If the sun is shining, then it is daytime,” “Francis Bacon wrote the works of Shakespeare.”

The negation of $P$, written $\sim P$, is true when $P$ is false and false when $P$ is true: “Francis Bacon did not write the works of Shakespeare.”

“$P$ and $Q$,” written $P \land Q$, is true when $P$ and $Q$ are both true, and false otherwise.

“$P$ or $Q$,” written $P \lor Q$, is true when one or both of $P$ and $Q$ is true, and false otherwise.

“$P$ implies $Q$,” also known as “if $P$, then $Q$,” and written $P \Rightarrow Q$, is true when $Q$ is true, and also when $P$ is false; otherwise, it is false.

“$P$ is equivalent to $Q$,” also known as “$P$ if and only if $Q$,” written $P \iff Q$, is true when $P$ and $Q$ are both true, and also when they are both false; otherwise, it is false.

The following statements are then true regardless of the truth value of $P$ and $Q$:

\[
\begin{align*}
(P \Rightarrow Q) \land P &\Rightarrow Q \quad \text{(modus ponens)} \\
(P \Rightarrow Q) \land \sim Q &\Rightarrow \sim P \quad \text{(modus tollens)} \\
(P \Rightarrow Q) \land (Q \Rightarrow R) &\Rightarrow (P \Rightarrow R) \\
(P \Rightarrow Q) &\iff (\sim Q \Rightarrow \sim P) \quad \text{(an implication is equivalent to its contrapositive)} \\
(P \Rightarrow Q) &\iff (Q \lor \sim P) \\
((P \Rightarrow Q) \land (Q \Rightarrow P)) &\iff (P \iff Q) \\
\sim (P \land Q) &\iff (\sim P \lor \sim Q) \quad \text{(the de Morgan laws)} \\
\sim (P \lor Q) &\iff (\sim P \land \sim Q)
\end{align*}
\]

You can prove these statements using truth tables.

On the other hand, the following statements can be false. They are the sources of common errors.

It’s a good exercise to find truth values of $P$ and $Q$ making these false.

\[
\begin{align*}
(P \Rightarrow Q) \land Q &\Rightarrow P \quad \text{(an implication may not work backwards)} \\
(P \Rightarrow Q) \land \sim P &\Rightarrow \sim Q \\
(P \Rightarrow Q) &\iff (Q \Rightarrow P) \quad \text{(an implication is not equivalent to its converse)} \\
(P \Rightarrow Q) &\iff (\sim P \Rightarrow \sim Q) \quad \text{(\ldots or its inverse)}
\end{align*}
\]
OUR NAIVE SET AXIOMS

Axiom of existence. There exists a set $A$.

Axiom of specification. For any set $A$ and any statement $P(x)$ involving an element $x$, there exists a subset $B$ of $A$ such that $x \in B$ if and only if $x \in A$ and $P(x)$ is true; it is denoted by $\{x \in A \mid P(x)\}$.

Axiom of singletons. For every set $A$, there is a set $\{A\}$ whose only element is $A$.

Axiom of unions. If $A$ and $B$ are sets, then there exists a set $A \cup B$, the union, such that $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

Axiom of intersections. If $A$ and $B$ are sets, then there exists a set $A \cap B$, the intersection, such that $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.

Axiom of powers. If $A$ is a set, there exists a set $\mathcal{P}(A)$, the power set, such that $x \in \mathcal{P}(A)$ if and only if $x$ is a subset of $A$.

Axiom of products. If $A$, $B$ are sets, there exists a set $A \times B$, the Cartesian product, whose elements are all the ordered pairs $(x, y)$, where $x \in A$ and $y \in B$.

AXIOMS OF THE REAL NUMBERS

There exists a set $\mathbb{R}$ having binary operations $+$ and $\cdot$, a relation $>$, and elements $0$ and $1 \in \mathbb{R}$, such that the following are true for all $x, y, z \in \mathbb{R}$:

Axiom of commutativity. $x + y = y + x$ and $x \cdot y = y \cdot x$.

Axiom of associativity. $(x + y) + z = x + (y + z)$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

Axiom of distributivity. $x \cdot (y + z) = x \cdot y + x \cdot z$.

Axiom of identity elements. $0 \neq 1$ and $0 + x = x = 1 \cdot x$.

Axiom of additive inverses. There exists $w \in \mathbb{R}$ such that $w + x = 0$; denote $w$ by $-x$.

Axiom of multiplicative inverses. If $x \neq 0$, then there exists $w \in \mathbb{R}$ such that $w \cdot x = 1$; denote $w$ by $1/x$.

Order axiom 1. If $x > 0$ and $y > 0$, then $x + y > 0$ and $x \cdot y > 0$.

Order axiom 2. If $x \neq 0$, then either $x > 0$ or $-x > 0$, but not both.

Order axiom 3. $0 \neq 0$.

Order axiom 4. If $x > y$, then $x + z > y + z$.

Axiom of completeness. For any subset $S \subset \mathbb{R}$ having an upper bound, that is, a number $b \in \mathbb{R}$ such that $b > x$ for all $x \in S$, there exists a least upper bound, that is, an upper bound $c$ satisfying $b > c$ for all other upper bounds $b$ for $S$. 