Mathematics V1207x
Honors Mathematics A

A SPEEDY REVIEW OF LOGIC

Letters P, Q, R denote *statements* which can be either true or false, but not both. For example, "All math teachers have red hair," "Crunchy-Wunchies are fortified with vitamins," "If the sun is shining, then it is daytime," "Francis Bacon wrote the works of Shakespeare."

The *negation* of P, written $\sim P$, is true when P is false and false when P is true: "Francis Bacon did not write the works of Shakespeare."

"P and Q," written $P \wedge Q$, is true when P and Q are both true, and false otherwise.

"P or Q," written $P \vee Q$, is true when one or both of P and Q is true, and false otherwise.

"P implies Q," also known as "if P, then Q," and written $P \Rightarrow Q$, is true when Q is true, and also when P is false; otherwise, it is false.

"P is equivalent to Q," also known as "P if and only if Q," written $P \Leftrightarrow Q$, is true when P and Q are both true, and also when they are both false; otherwise, it is false.

The following statements are then true regardless of the truth value of P and Q:

$((P \Rightarrow Q) \wedge P) \Rightarrow Q$ (modus ponens)

$((P \Rightarrow Q) \wedge \sim Q) \Rightarrow \sim P$ (modus tollens)

$((P \Rightarrow Q) \wedge (Q \Rightarrow R)) \Rightarrow (P \Rightarrow R)$

$(P \Rightarrow Q) \Leftrightarrow (\sim Q \Rightarrow \sim P)$ (an implication is equivalent to its contrapositive)

$(P \Rightarrow Q) \Leftrightarrow (Q \vee \sim P)$

$((P \Rightarrow Q) \wedge (Q \Rightarrow P)) \Leftrightarrow (P \Leftrightarrow Q)$

$\sim (P \wedge Q) \Leftrightarrow (\sim P \vee \sim Q)$

(the de Morgan laws)

$\sim (P \vee Q) \Leftrightarrow (\sim P \wedge \sim Q)$

You can prove these statements using truth tables.

On the other hand, the following statements can be *false*. They are the sources of common errors. It's a good exercise to find truth values of P and Q making these false.

$((P \Rightarrow Q) \wedge Q) \Rightarrow P$ (an implication may not work backwards)

$((P \Rightarrow Q) \wedge \sim P) \Rightarrow \sim Q$

$(P \Rightarrow Q) \Leftrightarrow (Q \Rightarrow P)$ (an implication is not equivalent to its converse)

$(P \Rightarrow Q) \Leftrightarrow (\sim P \Rightarrow \sim Q)$ (...or its inverse)

OUR NAIVE SET AXIOMS

Axiom of existence. There exists a set A.

Axiom of specification. For any set A and any statement $P(x)$ involving an element x, there exists a subset B of A such that $x \in B$ if and only if $x \in A$ and $P(x)$ is true; it is denoted by $\{x \in A \mid P(x)\}$.

Axiom of singletons. For every set A, there is a set $\{A\}$ whose only element is A.

Axiom of unions. If A and B are sets, then there exists a set $A \cup B$, the *union*, such that $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

Axiom of intersections. If A and B are sets, then there exists a set $A \cap B$, the *intersection*, such that $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.

Axiom of powers. If A is a set, there exists a set $\mathcal{P}A$, the *power set*, such that $x \in \mathcal{P}A$ if and only if x is a subset of A.

Axiom of products. If A, B are sets, there exists a set $A \times B$, the *Cartesian product*, whose elements are all the ordered pairs (x, y), where $x \in A$ and $y \in B$.

AXIOMS OF THE REAL NUMBERS

There exists a set $\mathbb{R}$ having binary operations $+$ and $\cdot$, a relation $>$, and elements 0 and $1 \in \mathbb{R}$, such that the following are true for all $x, y, z \in \mathbb{R}$:

Axiom of commutativity. $x + y = y + x$ and $x \cdot y = y \cdot x$.

Axiom of associativity. $(x + y) + z = x + (y + z)$ and $x \cdot (y \cdot z) = (x \cdot y) \cdot z$.

Axiom of distributivity. $x \cdot (y + z) = x \cdot y + x \cdot z$.

Axiom of identity elements. $0 \neq 1$ and $0 + x = x = 1 \cdot x$.

Axiom of additive inverses. There exists $w \in \mathbb{R}$ such that $w + x = 0$; denote w by $-x$.

Axiom of multiplicative inverses. If $x \neq 0$, then there exists $w \in \mathbb{R}$ such that $w \cdot x = 1$; denote w by $1/x$.

Order axiom 1. If $x > 0$ and $y > 0$, then $x + y > 0$ and $x \cdot y > 0$.

Order axiom 2. If $x \neq 0$, then either $x > 0$ or $-x > 0$, but not both.

Order axiom 3. $0 \not> 0$.

Order axiom 4. If $x > y$, then $x + z > y + z$.

Axiom of completeness. For any subset $S \subset \mathbb{R}$ having an upper bound, that is, a number $b \in \mathbb{R}$ such that $b > x$ for all $x \in S$, there exists a least upper bound, that is, an upper bound c satisfying $b > c$ for all other upper bounds b for S.