Mathematics V1207x
Honors Mathematics A

Assignment #9
Due November 24, 2015

Reading: Apostol, §§10.1–10.18, pp. 374–407.

***1.** If $\lim_{n\to\infty} a_n = A$, $\lim_{n\to\infty} b_n = B$, and $a_n \le b_n$ for all n, show that $A \le B$.

2. (Proposition 1) If $\{a_n\}$ is a sequence, prove that for any $k \in \mathbb{N}$, the series $\sum_{i=0}^{\infty} a_i$ converges if and only if $\sum_{i=k}^{\infty} a_i = \sum_{i=0}^{\infty} a_{i+k}$ does.

***3.** Given sequences $\{a_n\}$ and $\{b_n\}$ and a number $M \in \mathbb{N}$ such that $n \ge M$ implies $a_n = b_n$, prove that $\{a_n\}$ converges if and only if $\{b_n\}$ does.

***4.** (Proposition 2: "Agreement Test") Given sequences $\{a_n\}$ and $\{b_n\}$ and a number $M \in \mathbb{N}$ such that $n \ge M$ implies $a_n = b_n$, prove that the series $\sum_{i=1}^{\infty} a_i$ converges if and only if $\sum_{i=1}^{\infty} b_i$ does. [Of course it follows from the previous exercise, but with a little twist.]

***5.** (Proposition 3) Let $\{a_n\}$ be a sequence. Prove that if its series $\{\sum_{i=1}^{n} a_i\}$ converges, then the original sequence $\{a_n\}$ converges to 0.

6. Apostol §10.4 (p. 382) 1, 2, *29, 30, 31, 32.

7. Suppose $S \subseteq \mathbb{R}$. Show that $c = \sup S$ if and only if c is an upper bound for S and there exists a sequence $\{a_n\}$ with each $a_n \in S$ and $\lim_{n\to\infty} a_n = c$.

***8.** Show that a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if it "takes convergent sequences to convergent sequences," that is, whenever $\{x_n\}$ converges with $\lim_{n\to\infty} x_n = x$, then $\{f(x_n)\}$ converges with $\lim_{n\to\infty} f(x_n) = f(x)$.

9. Show that if $\{x_n\}$ converges, then so does $\{|x_n|\}$. Is the converse true? Give a proof or counterexample.

***10.** Let $\{a_n\}$ be a convergent sequence. Show that there exists $c \in \mathbb{R}$ such that for all n, $|a_n| \le c$. [Hint: any finite set has a maximum element.]

11. (a) Let $\{a_n\}$ be a sequence. Suppose that for all $c \in \mathbb{R}$, there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \ge N$ implies $|a_n| > c$. Prove that $\{a_n\}$ is divergent.

(b) Prove that if $|x| > 1$, then $\{x^n\}$ is divergent.

(c) Prove that if $|x| > 1$, then the geometric series $\sum_{n=0}^{\infty} x^n$ is divergent.

(d) Prove a counterpart to the Ratio Test: if $a_n > 0$ and $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = L > 0$, then $\sum_{n=0}^{\infty} a_n$ is divergent.