Mathematics V1207x
Honors Mathematics A

Assignment #5

Due October 16, 2015

Reading: Apostol, §2.18, pp. 120–124, and §§3.1–3.7, pp. 126–141.

***1.** What is $\sup\{\frac{n-1}{n} \mid n \in \mathbb{N} \setminus \{0\}\}$? What is $\inf\{\frac{n+1}{n} \mid n \in \mathbb{N} \setminus \{0\}\}$?

Prove your answers correct.

***2.** A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *even* if $f(-x) = f(x)$ for all x, and *odd* if $f(-x) = -f(x)$ for all x.

(a) Prove that if f is both odd and even, then $f(x) = 0$ for all x.

(b) Suppose f is integrable on every closed interval $[a, b]$, and let $g(x) = \int_0^x f(t)\,dt$. Prove that if f is odd, then g is even, and that if f is even, then g is odd.

***3.** Prove that $f : [a, b] \to \mathbb{R}$ is integrable if and only if for all $\varepsilon > 0$, there exist step functions $s, t : [a, b] \to \mathbb{R}$ such that $s \le f \le t$ and $\int_a^b (t - s)(x)\,dx < \varepsilon$.

(Hint: substitute $\varepsilon/2$ for ε in the approximation property of the sup.)

***4.** Prove that $\int_a^b x\,dx = (b^2 - a^2)/2$ in the following steps.

(a) Use the properties of integration to show that the general case is implied by the case where $a = 0$ and $b = 1$.

(b) Establish that $\int_0^1 x\,dx = 1/2$. (Hint: previous exercises may be useful.)

***5.** If $a \le c \le d \le b \in \mathbb{R}$, and $f : [a, b] \to \mathbb{R}$ is integrable on $[a, b]$, prove that it is integrable on $[c, d]$. (Hint: previous exercises may be useful.)

***6.** Suppose that f is integrable on $[a, b]$. Show that $|f|$ is integrable on $[a, b]$.

(Hint: if s and t are step functions such that $s \le f \le t$ and $\int (t - s) < \varepsilon$, and if a partition for both s and t is chosen, what step functions with this partition best approximate f above and below? Their definition will involve several cases.)

***7.** For all $x \in \mathbb{R}$ and $n \in \mathbb{N}$, define the *nth power* $x^n \in \mathbb{R}$ recursively (that is, inductively) by $x^0 = 1$ and $x^{n+1} = x \cdot x^n$.

(a) Using this definition, prove that the function $f(x) = x^n$ is monotone on $(-\infty, 0]$, and also on $[0, \infty)$.

(b) Prove that this function is integrable on any closed interval $[a, b]$.

(c) Prove that any polynomial function $g(x) = \sum_{i=0}^n c_i x^i$, where the c_i are constants, is integrable on $[a, b]$.

8. Apostol §2.19 (pp. 124–25) 19, 21.

9. Apostol §3.6 (p. 138) 3, *5, 6, 7, *8, 21.