Mathematics V1207x  
Honors Mathematics A  
Assignment #5  
Due October 16, 2015


*1. What is \( \sup \{ \frac{n-1}{n} \mid n \in \mathbb{N} \setminus \{0\} \} \)? What is \( \inf \{ \frac{n+1}{n} \mid n \in \mathbb{N} \setminus \{0\} \} \)?
Prove your answers correct.

*2. A function \( f : \mathbb{R} \rightarrow \mathbb{R} \) is said to be even if \( f(-x) = f(x) \) for all \( x \), and odd if \( f(-x) = -f(x) \) for all \( x \).
   (a) Prove that if \( f \) is both odd and even, then \( f(x) = 0 \) for all \( x \).
   (b) Suppose \( f \) is integrable on every closed interval \([a, b]\), and let \( g(x) = \int_0^x f(t) \, dt \).
       Prove that if \( f \) is odd, then \( g \) is even, and that if \( f \) is even, then \( g \) is odd.

*3. Prove that \( f : [a, b] \rightarrow \mathbb{R} \) is integrable if and only if for all \( \varepsilon > 0 \), there exist step functions \( s, t : [a, b] \rightarrow \mathbb{R} \) such that \( s \leq f \leq t \) and \( \int_a^b (t - s)(x) \, dx < \varepsilon \).
   (Hint: substitute \( \varepsilon/2 \) for \( \varepsilon \) in the approximation property of the sup.)

*4. Prove that \( \int_a^b x \, dx = \frac{(b^2 - a^2)}{2} \) in the following steps.
   (a) Use the properties of integration to show that the general case is implied by the case where \( a = 0 \) and \( b = 1 \).
   (b) Establish that \( \int_0^1 x \, dx = 1/2 \). (Hint: previous exercises may be useful.)

*5. If \( a \leq c \leq d \leq b \in \mathbb{R} \), and \( f : [a, b] \rightarrow \mathbb{R} \) is integrable on \([a, b]\), prove that it is integrable on \([c, d]\). (Hint: previous exercises may be useful.)

*6. Suppose that \( f \) is integrable on \([a, b]\). Show that \( |f| \) is integrable on \([a, b]\).
   (Hint: if \( s \) and \( t \) are step functions such that \( s \leq f \leq t \) and \( \int(t - s)(x) \, dx < \varepsilon \), and if a partition for both \( s \) and \( t \) is chosen, what step functions with this partition best approximate \( f \) above and below? Their definition will involve several cases.)

*7. For all \( x \in \mathbb{R} \) and \( n \in \mathbb{N} \), define the \( n \)th power \( x^n \in \mathbb{R} \) recursively (that is, inductively) by \( x^0 = 1 \) and \( x^{n+1} = x \cdot x^n \).
   (a) Using this definition, prove that the function \( f(x) = x^n \) is monotone on \((-\infty, 0]\),
       and also on \([0, \infty)\).
   (b) Prove that this function is integrable on any closed interval \([a, b]\).
   (c) Prove that any polynomial function \( g(x) = \sum_{i=0}^n c_i x^i \), where the \( c_i \) are constants,
       is integrable on \([a, b]\).
