Mathematics V1207x
Honors Mathematics A

Assignment #3
Due October 2, 2015

Reading: Apostol 1.1–1.10, pp. 48–63.

***1.** Let $S \subset \mathbb{N}$ be a subset of the natural numbers. An element $m \in S$ is called a *least element* if for all $n \in S$, $m \leq n$. (That is, m is inf S and is also an element of S.)

(a) Prove by induction that for all $k \in \mathbb{N}$, $\{\ell \in \mathbb{N} \,|\, \ell \leq k\}$ either contains a least element for S or does not contain any elements of S.

(b) Use (a) to prove the *well-ordering principle*: every nonempty subset of $\mathbb{N}$ has a least element.

2. Apostol §I 3.12 (pp. 28–29) #1, 3*, 4*, 6*, 7, 10, 11. (Hint: Use the well-ordering principle for #4.)

***3.** Suppose $S \subseteq \mathbb{R}$ and $c \in \mathbb{R}$. Let $cS = \{cx \,|\, x \in S\} \subseteq \mathbb{R}$. (Think of it as "stretching" S by a factor of c.)

(a) Show that if $c > 0$ and S is bounded above, then cS is bounded above.

(b) Show that if $c > 0$, then $\sup cS = c \sup S$.

(c) Give an example where $c < 0$ and $\sup cS \neq c \sup S$.

4. Suppose $S \subseteq \mathbb{R}$ and $t \in \mathbb{R}$. Show that $t = \sup S$ if and only if both of the following are true: (a) t is an upper bound for S, and (b) for all $\varepsilon > 0$, there exists $x \in S$ such that $x > t - \varepsilon$.

***5.** Suppose that $S, T \subseteq \mathbb{R}$, both S and T are nonempty and bounded above, and there is a bijective function $f : S \to T$ such that $x \geq f(x)$ for all $x \in S$. Show that $\sup S \geq \sup T$. Can you say more if you know $x > f(x)$ for all $x \in S$?

***6.** Prove that the Cartesian product of two finite sets is finite. (Hint: Consider first the case $S = \{1, \ldots, m\}$ and $T = \{1, \ldots, n\}$ and try induction on m. Then deduce the general case from this one.)

7. Prove that the intersection and union of two finite sets is finite. (You'll have to define a function to a subset of $\mathbb{N}$ and show that it is bijective. It might be convenient to use the well-ordering principle again.)

***8.** Prove that for all real $\epsilon > 0$ and all $x \in \mathbb{R}$, $|x| < \epsilon$ if and only if $-\epsilon < x < \epsilon$.

9. Apostol §I 4.9 (p. 43) #1bdfg, 1j*.