Mathematics V1207x
Honors Mathematics A

Assignment #12

Due Tuesday, December 15, 2015

Reading: Apostol §12.1–12.10, pp. 445–460, and §13.1–13.7, pp. 471–482.

In Apostol (5 points each):

§12.4 (p. 450) 5, *6, *7, 11a.

§12.8 (p. 456) 2, 3, *4, 10, *19, 20.

§12.11 (p. 460) 2, 7, 13, 17, 18, 19, 20.

§13.5 (p. 477) 5, 11, 12.

§15.5 (p. 555) 1, 2, 4, 5, 8, *9, *10, *11, *21, 24, 25, 29, 32.

Notes: Brevity is acceptable and encouraged. Apostol calls V_n what we called $\mathbb{R}^n$ in class. Two lines $\{X + tY \mid t \in \mathbb{R}\}$ and $\{X' + tY' \mid t \in \mathbb{R}\}$ in $\mathbb{R}^n$ are said to be *parallel* if Y is a scalar multiple of Y'.

Also do the following.

***1.** If W_1 and W_2 are subspaces of a vector space V, show that $W_1 \cap W_2$ is also a subspace. Give a specific example to show that $W_1 \cup W_2$ need not be a subspace.

2. Prove that $F : U \to V$ is linear if and only if for all $n \in \mathbb{N}$ and for all $X_1, \ldots, X_n \in U$ and all $c_1, \ldots, c_n \in \mathbb{R}$,

$$F\left(\sum_{i=1}^{n} c_i\, X_i\right) = \sum_{i=1}^{n} c_i\, F(X_i).$$

***3.** Prove that if $F : U \to V$ is linear, then the kernel, $\ker F$, is a subspace of U and the image, $\operatorname{im} F$, is a subspace of V.

***4.** Let $F : U \to V$ be linear. Prove that it is injective if and only if $\ker F = \{\vec{0}\}$.

5. If U, V are vector spaces, prove that the set $\mathcal{L}(U, V)$ of linear maps $U \to V$ is a subspace of the vector space $\mathcal{F}(U, V)$ of all maps $U \to V$.

***6.** Prove that the composition of two linear maps is linear.

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7. (a) If $G : V \to W$ is linear and fixed, prove that the map $L_G : \mathcal{L}(U, V) \to \mathcal{L}(U, W)$ given by $L_G(F) = G \circ F$ is linear.

(b) If $F : U \to V$ is linear and fixed, prove that the map $R_F : \mathcal{L}(V, W) \to \mathcal{L}(U, W)$ given by $R_F(G) = G \circ F$ is linear.

***8.** Given a matrix $A \in M_{m\times n}$, show that the map $T_A : \mathbb{R}^n \to \mathbb{R}^m$ given by

$$T_A(x_1, \ldots, x_n) = \left(\sum_{j=1}^{n} A_{1j}\, x_j, \sum_{j=1}^{n} A_{2j}\, x_j, \ldots, \sum_{j=1}^{n} A_{mj}\, x_j \right)$$

is linear.

***9.** Show that the map $T : M_{m\times n} \to \mathcal{L}(\mathbb{R}^m, \mathbb{R}^n)$ given by $T(A) = T_A$ is linear.