Mathematics V1207x
Honors Mathematics A

Assignment #11
Due Tuesday, December 8, 2015

Reading: Apostol, §§11.8–11.10, pp. 431–436.

Problems **4** and **5** are the remarkable results referred to in the lecture as Theorems 1 and 2.

Hint: Do the problems in order.

***1.** Let m be any positive integer.

(a) Show that for all $y \in \mathbb{R}$, $y \geq m$ implies $2y - \frac{m}{y} \geq 1$.

(b) Show there exists $c \in \mathbb{R}$ such that $z \geq m$ implies $z^2 - m \ln z \geq z + c$.

(c) Show that $w : [m, \infty) \to \mathbb{R}$ given by $w(z) = z^2 - m \ln z$ is an increasing and unbounded function of z.

(d) Show that for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $x \in \mathbb{R}$,

$$0 < |x| < \delta \Rightarrow -\frac{1}{x^2} - m \ln |x| < \ln \epsilon.$$

***2.** Apostol §7.17 (p. 304) *31.

***3.** (a) Let

$$g(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0, \\ 0 & \text{if } x \leq 0. \end{cases}$$

Show that $g : \mathbb{R} \to \mathbb{R}$ is a smooth function which is positive on $(0, \infty)$.
[Be careful about smoothness at 0.] Also sketch its graph.

(b) With g as above, let $h(x) = g(1 + x)g(1 - x)$.
Then let $A = \int_{-1}^{1} h(t)\,dt$, and let $F(x) = \frac{1}{A} \int_{-1}^{x} h(t)\,dt$.
Show that $F : \mathbb{R} \to \mathbb{R}$ is a smooth function such that $F(x) = 0$ for $x < -1$ and $F(x) = 1$ for $x > 1$. Also sketch its graph.

(c) Show there exists a smooth function $\phi : \mathbb{R} \to \mathbb{R}$ such that $0 \leq \phi(x) \leq 1$ for all x, $\phi(x) = 1$ for $x \in (-1, 1)$, and $\phi(x) = 0$ for $x \notin [-3, 3]$. Such a function ϕ is called a "bump function". Sketch it too.

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***4.** If $g, h : (a, b) \to \mathbb{R}$ are two smooth functions and c, d are distinct points in (a, b), show that there exists $\delta > 0$ and a smooth function $f : (a, b) \to \mathbb{R}$ such that for all $x \in (a, b)$,

$$|x - c| < \delta \Rightarrow f(x) = g(x)$$

and

$$|x - d| < \delta \Rightarrow f(x) = h(x).$$

[Hint: multiply g by a function closely related to ϕ. You should be able to take $\delta = |c - d|/6$.]

***5.** Suppose that $f, g : (a, b) \to \mathbb{R}$ are two analytic functions, and that there exist $c \in (a, b)$ and $\delta > 0$ such that for all $x \in (a, b)$, $|x - c| < \delta$ implies $f(x) = g(x)$. Show that $f = g$. More colloquially, if two analytic functions on an open interval agree on some open subinterval, then they agree everywhere. [Hint: let

$$U = \{x \in [c, b) \mid f(y) = g(y) \text{ for all } y \in [c, x]\}.$$

Show that this has a supremum, and study $f - g$ near the supremum.]