Mathematics V1207x
Honors Mathematics A

Assignment #10
Due Tuesday, December 1, 2015

Reading: Apostol, §§10.21–23, pp. 411–420, and §§11.1–11.5, pp. 422–427.

1. Apostol §11.7 (pp. 430–431) 1, 2, 3, 6, 7, 8, 17, *18.

[On the unstarred problems, if testing the endpoints proves difficult, don't bother.]

2. Let $\{a_n\}$ be an increasing convergent sequence with limit a. Show that for each n, $a_n \leq a$.

***3.** Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of integrable functions converging uniformly to $f : [a,b] \to \mathbb{R}$. Prove that f is integrable and

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx.$$

[Don't assume the f_n are continuous. You might want to use #3 from assignment 5.]

***4.** Let $f_n(x) = \dfrac{x}{1+nx^2}$.

(a) Find $f(x) = \lim_{n\to\infty} f_n(x)$, and $g(x) = \lim_{n\to\infty} f_n'(x)$.

(b) Prove that for all $x \in \mathbb{R}$, $|f_n(x)| \leq \sqrt{1/n}$. [Hint: find the local extrema.] Do the f_n converge uniformly? Why or why not?

(c) Prove f is differentiable at every $x \in \mathbb{R}$. For what x is $f'(x) = g(x)$?

5. What "theorem" is disproved by the previous problem?

6. Let $I \subseteq \mathbb{R}$ be any interval, and let $\{f_n\}$ be a sequence of functions $I \to \mathbb{R}$. Prove that if $f_n \to f$ uniformly for some f, and if each f_n is bounded, then the sequence is *uniformly bounded*, that is, there exists a single $M \in \mathbb{R}$ such that for all $n \in \mathbb{N}$ and $x \in I$, $|f_n(x)| \leq M$.

7. If f_n and g_n are sequences of bounded functions on an interval I, and $f_n \to f$ and $g_n \to g$, both uniformly, prove that

(a) $cf_n \to cf$ uniformly for any $c \in \mathbb{R}$;

*(b) $f_n + g_n \to f + g$ uniformly;

(c) $f_n g_n \to fg$ uniformly. [Harder. Use uniform boundedness and some ingenuity.]

***8.** Prove that the series below is everywhere convergent to a continuous function that can be integrated term by term:

$$\sum_{n=0}^{\infty} \frac{1}{2^n + x^{2n}}.$$

(To say that it can be integrated *term by term* means that the integral of the infinite sum is the infinite sum of the integrals of the individual terms.)