Mathematics V1207x
Honors Mathematics A

Answers to Final Exam
December 21, 2015

1. True: for all $z \in U$, there exists $y \in T$ such that $z = g(y)$, and $x \in S$ such that $y = f(x)$, so $z = g(f(x)) = g \circ f(x)$.

2. True: differentiable implies continuous, which implies integrable.

3. False: indefinite integrals are continuous, but this isn't.

4. True: if $|a_n| \leq B$, then $\sum |a_n| 2^n/n!$ converges by comparison to $\sum B2^n/n!$, which converges by the ratio test. (In fact, the sum is Be^2.) The original series hence converges absolutely, so it converges.

5. False: $G(2\,\mathrm{id}_V) = (2\,\mathrm{id}_V) \circ (2\,\mathrm{id}_V) = 4\,\mathrm{id}_V \neq 2\,\mathrm{id}_V = 2\,\mathrm{id}_V \circ \mathrm{id}_V = 2\,G(\mathrm{id}_V)$.

6. We say $c \in [a, b]$ is an *absolute maximum* of f if for all $x \in [a, b]$, $f(x) \leq f(c)$; we say it is a *relative maximum* of f if there exists $\delta > 0$ such that for all $x \in [a, b]$, $|x - c| < \delta \implies f(x) \leq f(c)$.

7. First fundamental theorem: Suppose $f : [a, b] \to \mathbb{R}$ is is integrable and $c \in [a, b]$. Let $g(x) = \int_c^y f(y)\,dy$. If f is continuous at $x \in (a, b)$, then g is differentiable at x and $g'(x) = f(x)$. Second fundamental theorem: Suppose g is an antiderivative of a function f continuous on some interval I. Then for any a, b in that interval, $g(b) - g(a) = \int_a^b f(x)\,dx$.

8. Let $f_n : I \to \mathbb{R}$ be a sequence of functions defined on an interval I. Suppose there exists a sequence of numbers $M_n \in \mathbb{R}$ such that $\sum_{n=0}^{\infty} M_n$ converges and for all $x \in I$, $|f_n(x)| \leq M_n$. Then the series of functions $\sum_{n=0}^{\infty} f_n$ converges uniformly (and absolutely) on I.

9. For all $X, Y \in \mathbb{R}^n$, $|X \cdot Y| \leq \|X\|\,\|Y\|$.

10. Note that $|f(0)| \leq |3 \cdot 0| = 0$, so $f(0) = 0$. Given $\epsilon > 0$, take $\delta = \epsilon/3$. Then for all $x \in \mathbb{R}$, $|x - 0| < \delta \implies |f(x)| \leq |3x| = 3|x| < 3\delta = 3\epsilon/3 = \epsilon$, so $\lim_{x \to 0} f(x) = 0 = f(0)$ and f is continuous at 0.

11. Let S be the image of f. By the extreme value theorem, f has an absolute minimum x and absolute maximum y. Let $c = f(x)$ and $d = f(y)$. By the definitions of absolute minimum and maximum, $S \subset [c, d]$. Given any $e \in [c, d]$, by the intermediate value theorem there exists $z \in [a, b]$ such that $e = f(z)$. Hence $[c, d] \subset S$ as well.

12. First note that this is true for $x = 1$ as $\ln 1 = 0$. By the comparison theorem for integrals $0 = \int_1^x 0\,dt \leq \int_1^x dt/t = \ln x$ for $x \geq 1$, so $1 \leq 1 + \ln x$ for $x \geq 1$, and by the comparison theorem again, $x - 1 = \int_1^x 1\,dx \leq \int_1^x (1 + \ln x)\,dx = x \ln x$.

Alternative: the function $f(x) = x \ln x - x + 1$ has derivative $\ln x > 0$ for $x > 1$, so it is strictly increasing on $[1, \infty)$, but $f(1) = 0$, so for $x > 1$, $f(x) > 0$.

Alternative: applying the mean-value theorem to ln on $[1, x]$, there exists $z \in (1, x)$ such that $1/z = \ln'(z) = \frac{\ln x - \ln 1}{x - 1} = (\ln x)/(x - 1)$. Hence $(x - 1)/(\ln x) = z \leq x$ for $x > 1$.

13. Substituting x^2 for x in the geometric series, we find that $f(x) = \sum_{n=0}^{\infty} x^{2n}$ for $|x^2| < 1$, that is, for $|x| < 1$. Since this is a power series centered at 0 converging to $f(x)$, by a theorem from class it must agree with the Taylor series at 0, namely $\sum_{m=0}^{\infty} \frac{f^{(m)}(0)x^m}{m!}$. The coefficient of x^{600} is therefore $1 = f^{(600)}(0)/600!$, so $f^{(600)}(0) = 600!$.

14. Since $\sum_{n=0}^{\infty} a_n$ is convergent, $\lim_{n\to\infty} a_n = 0$ by A9#5 = Proposition 3. Take $\epsilon = 1$; then there exists $N \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N \implies |a_n| < 1$, hence $a_n < 1$ since all terms are nonnegative. Then $0 \leq a_n\, b_n \leq b_n$, so $\sum_{n=N}^{\infty} a_n\, b_n$ converges by comparison with $\sum_{n=N}^{\infty} b_n$, so $\sum_{n=0}^{\infty} a_n\, b_n$ converges by A9#2 = Proposition 1.

Challenge problem: take $a_n = b_n = (-1)^n/\sqrt{n}$. Then $\sum_{n=1}^{\infty} a_n$ is convergent by the alternating series test, but $\sum_{n=1}^{\infty} a_n^2$ is the harmonic series.

15. If $S, T : V \to W$ are linear and $X, Y \in V$, then $(S+T)(X+Y) = S(X+Y) + T(X+Y) = S(X) + S(Y) + T(X) + T(Y) = S(X) + T(X) + S(Y) + T(Y) = (S+T)(X) + (S+T)(Y)$. Also, if S, T are as before, $X \in V$, and $c \in \mathbb{R}$, then $(S+T)(cX) = S(cX) + T(cX) = cS(X) + cT(X) = c(S(X) + T(X)) = c((S+T)(X)) = (c(S+T))(X)$. Hence $S+T$ is linear.

If $S : V \to W$ is linear, $d \in \mathbb{R}$, and $X, Y \in V$, then $(dS)(X+Y) = d(S(X+Y)) = d(S(X) + S(Y)) = d(S(X)) + d(S(Y)) = (dS)(X) + (dS)(Y)$. Also, if S, d are as before, $X \in V$, and $c \in \mathbb{R}$, then $(dS)(cX) = d(S(cX)) = d(c(S(X)) = (dc)(S(X)) = c(dS)(X)$. Hence dS is linear.

16. Observe first that for smooth f, g and for $t \in \mathbb{R}$, we have $(f+g)^{(n)}(0) = f^{(n)}(0) + g^{(n)}(0)$ and $(tf)^{(n)}(0) = t\, f^{(n)}(0)$, both by induction on n: the case $n = 0$ is true by definition, and the induction step follows directly from the linearity of the derivative.

(a) We know that V is a nonempty subset of the vector space $\mathcal{F}(\mathbb{R}, \mathbb{R})$. If f and g are analytic, then the Taylor series of $f + g$ at c is the sum of those for f and g, and the Taylor series of tf at c is t times that of f, by the observations above. Hence, if the Taylor series of f and g, respectively, at c converge to f and g with positive radii of convergence R and R', then their sum converges with radius of convergence at least $\min(R, R')$, and that of tf converges with radius of convergence R. Hence $f + g$ and tf are analytic [this was actually stated in class so you could just quote it], so V is a subspace.

(b) The linearity of the map G amounts to showing $G(f+g) = G(f) + G(g)$ and $G(tf) = tG(f)$, which follows directly from the observations at the beginning.

(c) Yes, it is injective as it has kernel $\{0\}$: if $G(f) = 0$, then the Taylor series of f at 0 is 0, and then $f = 0$ by A11#5.

Challenge problem: all similar, but it is not injective as $f(x) = e^{-1/x^2}$ if $x \neq 0$, $f(x) = 0$ is $x = 0$ is now in the kernel of G.