# Mathematics V1207x
# Honors Mathematics A

**Final Examination**
December 21, 2015

Turn off all electronic devices.

Brief visits to the men's or women's room (just out the door to your left) are OK, but one at a time only, and you must tell me before you go.

Write your name, "Honors Math A, Prof. Thaddeus," and the number of blue books on the cover of each blue book.

Write your name on the attendance sheet when it comes around.

Write all answers and work in your blue books. Do not hand in this sheet.

On each page you use, write the number of the problem *in a circle in the margin.*

You may do more than one problem on a single page. Just put all numbers by their problems.

You may do the problems out of order, but this is discouraged as it can lead to misgrading.

When there is any doubt, state briefly but clearly what statements from Apostol, lecture, or assignments you are using.

You may use, without comment, facts from logic and high-school algebra.

In grading the exams, I will emphasize accuracy, brevity, and clarity. Aim for all three.

A perfect score is 124 points. Good luck!

**PART A:** True/False. Decide whether the given statement is true or false, and give a brief reason for your answer (sketch of proof or counterexample). Keep it simple! 6 points each: 2 for answer, 4 for reason.

**1.** If $f : S \to T$ and $g : T \to U$ are surjective, then $g \circ f : S \to U$ is surjective.

**2.** If $f : \mathbb{R} \to \mathbb{R}$ is everywhere differentiable, then it is integrable on every $[a, b]$.

**3.** The greatest-integer function $f(x) = [x]$ is the indefinite integral of some $g : \mathbb{R} \to \mathbb{R}$.

**4.** If $a_0, a_1, a_2, \ldots$ is a bounded sequence, then the series $\displaystyle\sum_{n=0}^{\infty} \frac{a_n 2^n}{n!}$ converges.

**5.** For a vector space $V$, if $\mathcal{L}(V, V)$ denotes the space of linear functions $V \to V$, then $G : \mathcal{L}(V, V) \to \mathcal{L}(V, V)$ given by $G(T) = T \circ T$ is itself a linear function.

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**PART B:** Statements. State in a complete sentence including all hypotheses. 6 points each.

**6.** Define (a) an absolute maximum and (b) a relative maximum of $f : [a, b] \to \mathbb{R}$.

**7.** State either of the fundamental theorems of calculus.

**8.** State the Weierstrass $M$-test.

**9.** State the Cauchy-Schwarz inequality.

**PART C:** Proofs. Give complete proofs of as many as possible. 10 points each.

**10.** Suppose $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying $|f(x)| \leq |3x|$ for all $x \in \mathbb{R}$.
Prove that $f$ is continuous at 0.

**11.** The *image* of any function $f : S \to T$ is defined to be $\{f(x) \,|\, x \in S\} \subset T$.
If $f : [a, b] \to \mathbb{R}$ is continuous, prove that its image is $[c, d]$ for some $c, d \in \mathbb{R}$.

**12.** For $x \geq 1$, prove that $x - 1 \leq x \ln x$.

**13.** Compute the 600th derivative of $f(x) = 1/(1 - x^2)$ at 0. Prove your answer correct.
You may express the answer in terms of factorials.

**14.** Suppose $\displaystyle\sum_{n=0}^{\infty} a_n$ and $\displaystyle\sum_{n=0}^{\infty} b_n$ are convergent series with all $a_n \geq 0$ and $b_n \geq 0$.

Prove that $\displaystyle\sum_{n=0}^{\infty} a_n \, b_n$ is convergent.

Challenge problem (not for credit): give a counterexample if not all $a_n$ and $b_n$ are $\geq 0$.

**15.** For vector spaces $V$ and $W$, prove that the set $\mathcal{L}(V, W)$ of linear functions $V \to W$ is a subspace of the space $\mathcal{F}(V, W)$ of all functions $V \to W$.

**16.** Let $V$ be the set of all analytic functions $\mathbb{R} \to \mathbb{R}$.

(a) Prove that $V$ is a vector space.

(b) Prove that the map $G : V \to \mathcal{F}(\mathbb{N}, \mathbb{R})$ given by $G(f) = \{f^{(n)}(0)\}$ is linear.

(c) Is $G$ injective? Why or why not?

Challenge problem (not for credit): same thing with "analytic" replaced by "smooth."