1. If $A \in M_{m,n}(\mathbb{R})$ and $B \in M_{n,p}(\mathbb{R})$, prove that $(AB)^T = B^T A^T$.

2. For $A$ and $B$ as above, prove that rank $AB \leq$ rank $A$ and also that rank $AB \leq$ rank $B$.

3. (a) Find the inverse of the matrix $A = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix}$.

(b) If $\bar{c} = \begin{pmatrix} 5 \\ 1 \\ 3 \end{pmatrix}$, find all solutions $\bar{x}$ of the inhomogeneous system $A\bar{x} = \bar{c}$.

4. (a) If $A$ is a square matrix with real entries, prove that $AA^T$ is symmetric.

(b) Show that the symmetric matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}$ cannot be expressed as $AA^T$ for any square $A$ with real entries.

5. (a) Prove that for $a, b \in \mathbb{C}$, the matrix $A = \begin{pmatrix} 0 & a \\ b & 0 \end{pmatrix}$ is diagonalizable if and only if $a$ and $b$ are both zero or both nonzero.

(b) Moreover, $\mathbb{C}^2$ has an orthonormal basis of $A$-eigenvectors if and only if $|a| = |b|$.

6. Find an orthonormal basis for the linear span of $(2, 1, 2)$ and $(1, 1, 1)$ in $\mathbb{R}^3$.

7. (a) If $A$ is unitary, prove that its eigenvalues have absolute value 1.

(b) If $A$ is unitary, prove that it is diagonalizable (over $\mathbb{C}$).