Except for concatenation, all the basic properties of the one-variable Riemann integral (such as linearity) carry over straightforwardly to multiple integrals, both in their statements and in their proofs. You may assume them. An analogue of concatenation is described below.


2. Let \( f : [0, \infty) \to \mathbb{R} \) be a function continuous except at one \( a \in \mathbb{R} \), and such that \( f(t) = 0 \) for all \( t > a \). For \( x \in \mathbb{R}^2 \setminus (0, 0) \), define \( g(x) = \sum_{i=1}^{\infty} f(i\|x\|) \). Show that this is well-defined and integrable on any closed rectangle not containing \((0, 0)\).

3. Give a counterexample to show that uniform continuity is false for a continuous function on an open rectangle.

*4. Suppose that \( T \) is of graph type in \( \mathbb{R}^{n-1} \), \( f : T \to \mathbb{R} \) is continuous, and (for some \( i \))

\[
P = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i = f(x_1, \ldots, \hat{x_i}, \ldots)\}.
\]

Show that for any continuous function \( \phi : P \to \mathbb{R} \), \( \int\int_P \phi = 0 \). Sketch in \( \mathbb{R}^2 \) or \( \mathbb{R}^3 \).

Hint: \( P \) is of graph type.

5. (a) Show that if \( S \) and \( T \) are disjoint subsets of \( \mathbb{R}^n \), and if \( \phi : \mathbb{R}^n \to \mathbb{R} \) is a bounded function such that \( \int\int_S \phi \) and \( \int\int_T \phi \) exist, then \( \int\int_{S \cup T} \phi \) exists and equals the sum of the previous two integrals. Hint: use linearity for multiple integrals.

(b) Same as (a), except that \( S \) and \( T \) now intersect in a set \( P \) as in 4 above. You can regard this as a concatenation theorem for multiple integrals.

6. Let \( a, b, c, d \) be continuous functions \( [0, 1] \to \mathbb{R} \) with \( a < b \) and \( c < d \), and let \( Q(t) \) be the rectangle \([a(t), b(t)] \times [c(t), d(t)]\). Show that if \( f : \mathbb{R}^2 \to \mathbb{R} \) is a continuous function, then \( \int\int_{Q(t)} f \) is a continuous function of \( t \).

7. Apostol §11.22 (pp. 385–7) *1ad, 2, 4, *5, 6, 7, 8ab. In the problems calling for a region bounded by a Jordan curve, you may substitute a region of graph type and its boundary curve.