Notation: for $x \in \mathbb{R}$, $[x]$ denotes the greatest integer less than or equal to $x$.

1. State the sign-preserving property of continuous functions.

2. Give an example of a subset of $\mathbb{R}$ that: (a) contains its supremum; (b) does not contain its supremum; (c) has no supremum.

3. Write the negation of the following statement in terms of $\sim P(x)$ and $\sim Q(x, y)$:

   $$\forall x \in S \left( P(x) \text{ and } \exists y \in T \mid Q(x, y) \right).$$

4. Prove by induction that for all $n \in \mathbb{N}$,

   $$\int_0^n [t]^2 \, dt = n(n - 1)(2n - 1)/6.$$

5. A function $f : \mathbb{R} \to \mathbb{R}$ is said to be periodic if for all $x \in \mathbb{R}$ and all $n \in \mathbb{Z}$, $f(x) = f(x + n)$. Which of the following are periodic? Why or why not?

   (a) $f(x) = x - [x];$ (b) $g(x) = 1 - x^2;$(c) $h(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$

6. Suppose $f : \mathbb{R} \to \mathbb{R}$ is periodic and integrable on all intervals $[a, b] \subseteq \mathbb{R}$, and let $g(x)$ be the indefinite integral $\int_0^x f(t) \, dt$. Prove that $g$ is periodic if and only if $\int_0^n f(t) \, dt = 0$ for all $n \in \mathbb{Z}$.

7. Suppose $f$ is periodic and continuous at every $x \in \mathbb{R}$. Prove that it is bounded.
1. State the completeness axiom of the real numbers.

2. If $f$ and $g$ are both even functions, does $g \circ f$ have to be even? Give a proof or counterexample. What if they are both odd?

3. Prove that if $f : [a, b] \to \mathbb{R}$ is monotone, then it is bounded.

4. Let $f$ and $g$ be continuous functions from $[a, b]$ to $\mathbb{R}$ such that $f(a) < g(a)$ and $f(b) > g(b)$. Prove that there exists $c \in (a, b)$ such that $f(c) = g(c)$.

5. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = 0$ if $x$ is irrational, $f(x) = x^2$ if $x$ is rational. Show that $f$ is continuous at 0. (Don’t be scared of this function; just write out the definition of the relevant limit.)

6. Show that
   \[
   \lim_{x \to 0} x \sin(1/x) = 0.
   \]

7. Evaluate the following integral without doing any computation. Justify your answer.
   \[
   \int_{-2}^{2} \frac{x^3 (x^2 + 7)}{x^4 + 12} \, dx
   \]