1. Apostol §11.7 (pp. 430–431) 17, *18.

2. Let \( \{a_n\} \) be an increasing convergent sequence with limit \( a \). Show that for each \( n \), \( a_n \leq a \).

*3. Let \( \{a_n\} \) be a convergent sequence. Show that there exists \( c \in \mathbb{R} \) such that for all \( n \), \( |a_n| \leq c \). [Hint: any finite set has a maximum element.]

*4. Let \( f_n : [a, b] \to \mathbb{R} \) be a sequence of integrable functions converging uniformly to \( f : [a, b] \to \mathbb{R} \). Prove that \( f \) is integrable and

\[
\lim_{n \to \infty} \int_a^b f_n(x) \, dx = \int_a^b f(x) \, dx.
\]

[Don’t assume the \( f_n \) are continuous. You might want to use #2 from assignment 5.]

*5. Let \( f_n(x) = \frac{x}{1 + nx^2} \).

(a) Find \( f(x) = \lim_{n \to \infty} f_n(x) \), and \( g(x) = \lim_{n \to \infty} f'_n(x) \).

(b) Prove that for all \( x \in \mathbb{R} \), \( |f_n(x)| \leq \sqrt{1/n} \). [Hint: find the local extrema.]

Do the \( f_n \) converge uniformly? Why or why not?

(c) Prove \( f \) is differentiable at every \( x \in \mathbb{R} \). For what \( x \) is \( f'(x) = g(x) \)?

6. What “theorem” is disproved by the previous problem?

7. Let \( I \subseteq \mathbb{R} \) be any interval, and let \( \{f_n\} \) be a sequence of functions \( I \to \mathbb{R} \). Prove that if \( f_n \to f \) uniformly for some \( f \), and if each \( f_n \) is bounded, then the sequence is uniformly bounded, that is, there exists a single \( M \in \mathbb{R} \) such that for all \( n \in \mathbb{N} \) and \( x \in I \), \( |f_n(x)| \leq M \).

8. If \( f_n \) and \( g_n \) are sequences of bounded functions on an interval \( I \), and \( f_n \to f \) and \( g_n \to g \), both uniformly, prove that

(a) \( cf_n \to cf \) uniformly for any \( c \in \mathbb{R} \);

*(b) \( f_n + g_n \to f + g \) uniformly;

(c) \( f_n g_n \to fg \) uniformly. [Harder. Use uniform boundedness and some ingenuity.]

*9. Prove that the series below is everywhere convergent to a continuous function that can be integrated term by term:

\[
\sum_{n=0}^{\infty} \frac{1}{2^n + x^{2n}}.
\]