Mathematics V1208y
Honors Mathematics B

Practice Midterm Exam

March 9, 2016

1. State the theorem on the existence and uniqueness of the determinant.

2. (a) If A is a square matrix with real entries, prove that AA^T is symmetric.

(b) Show that the symmetric matrix $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 3 & 1 \end{pmatrix}$ cannot be expressed as AA^T for any square A with real entries.

3. Let A be a square matrix with complex entries.

(a) Prove that $\det A^* = \overline{\det A}$ by reduction by minors.

(b) If λ is an eigenvalue of A, show that $\overline{\lambda}$ is an eigenvalue of A^*.

4. Let $V \subset \mathcal{F}([-1,1], \mathbb{R})$ be the space of continuous functions $[-1,1] \to \mathbb{R}$, with Euclidean inner product given by

$$\langle f, g \rangle = \int_{-1}^{1} f(x)g(x)\, dx.$$

Find an orthonormal basis for the subspace spanned by $1, x, x^2$.

(Apply the Gram-Schmidt process to this subspace.)

5. (a) Find the inverse of the matrix $B = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 0 & 1 \\ 2 & 0 & 1 \end{pmatrix}$.

(b) If $\vec{c} = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}$, find all solutions $\vec{x}$ of the inhomogeneous system $B\vec{x} = \vec{c}$.

6. Let A be a square matrix with real entries. If λ is a complex eigenvalue of A, prove that $\bar{\lambda}$ is also an eigenvalue of A.

7. (a) If X, Y are vectors in a real Euclidean space, prove that $\langle X, Y \rangle = 0$ if and only if $\|X - Y\| = \|X + Y\|$.

(b) Give a counterexample showing that this is false in a complex Hermitian space.

ANSWERS OVERLEAF...

Answers to Practice Midterm

1. There exists an unique $\det : M_{n\times n} \to \mathbb{R}$ which, when expressed as a function $\det(\vec{a}_1, \ldots, \vec{a}_n)$ of the rows $\vec{a}_1, \ldots, \vec{a}_n$ of $A \in M_{n\times n}$, satisfies the following:

D1: For any $i \in \{1, \ldots, n\}$ and any $c \in \mathbb{R}$, $\det(\vec{a}_1, \ldots, c\vec{a}_i, \ldots, \vec{a}_n) = c\det(\vec{a}_1, \ldots, \vec{a}_n)$.

D2: For any $i \in \{1, \ldots, n\}$ and any $\vec{b}_i \in \mathbb{R}^n$,

$$\det(\vec{a}_1, \ldots, \vec{a}_i + \vec{b}_i, \ldots, \vec{a}_n) = \det(\vec{a}_1, \ldots, \vec{a}_i, \ldots, \vec{a}_n) + \det(\vec{a}_1, \ldots, \vec{b}_i, \ldots, \vec{a}_n).$$

D3: If two distinct rows are equal, say $\vec{a}_i = \vec{a}_j$ for $i \neq j$, then $\det(\vec{a}_1, \ldots, \vec{a}_n) = 0$.

D4: $\det I_n = 1$.

2. (a) $(AA^T)^T = (A^T)^T A^T = AA^T$, so this is symmetric.

(b) $\det(AA^T) = \det A \det A^T = (\det A)^2 \geq 0$, but this matrix has determinant -8.

3. (a) $A^* = \overline{A^T}$ and $\det A^T = \det A$, so it suffices to show $\det \overline{A} = \overline{\det A}$. Proof by induction on n, where A is $n \times n$. Obvious for $n = 1$. If true for $n-1$, $\det \overline{A} = \sum_i (-1)^i \overline{a}_{i1} \det \overline{A}_{i1} = \sum_i (-1)^i \overline{a}_{i1} \overline{\det A_{i1}} = \overline{\det A}$, the 2nd equality by induction, the 3rd since conjugation commutes with taking sums and products.

(b) If $\det(\lambda I - A) = 0$, then by (a), $\det(\overline{\lambda} I - A^*) = \det(\overline{\lambda} I^* - A^*) = \det(\lambda I - A)^* = \overline{0} = 0$.

4. $\|1\|^2 = \langle 1, 1\rangle = 2$, so let $u_1 = 1/\|1\| = 1/\sqrt{2}$. Then $\langle u_1, x\rangle = 0$, so let $u_2 = x/\|x\| = x/\sqrt{2/3} = \sqrt{3}x/\sqrt{2}$. Finally, $\langle u_1, x^2\rangle = \sqrt{2}/3$, and $\langle u_2, x^2\rangle = 0$, so let u_3 be $x^2 - \sqrt{2}/3\, u_1$ divided by its own norm, which is $(3\sqrt{5})/(2\sqrt{2})(x^2 - 1/3)$.

5. (a) Elementary row operations on the augmented matrix $(B|I)$ lead to $(I|B^{-1})$ where

$$B^{-1} = \begin{pmatrix} 0 & -1 & 1 \\ 1 & -6 & 3 \\ 0 & 2 & -1 \end{pmatrix}.$$

(On a real exam, you must show your work! Also, it's wise to check $BB^{-1} = I$.)

(b) Multiplying both sides by B^{-1} shows that the unique solution is $B^{-1}\vec{c} = (3, 4, -1)$.

6. Proof 1: let $\chi(t) = \det(tI - A)$ be the characteristic polynomial of A. Then clearly χ has real coefficients, that is, $\chi(t) = c_0 + c_1 t + c_2 t^2 + \cdots + t^n$ with $c_i \in \mathbb{R}$. Now λ is an eigenvalue if and only if $\chi(\lambda) = 0$, but then $\chi(\bar{\lambda}) = c_0 + c_1\bar{\lambda} + c_2\bar{\lambda}^2 + \cdots + \bar{\lambda}^n = \bar{c}_0 + \bar{c}_1\bar{\lambda} + \bar{c}_2\bar{\lambda}^2 + \cdots + \bar{\lambda}^n = \bar{0} = 0$, so $\bar{\lambda}$ is also an eigenvalue. Proof 2: say $X = (x_1, \ldots x_n)$ is the eigenvector, $AX = \lambda X$. Let $\bar{X} = (\bar{x}_1, \ldots, \bar{x}_n)$. Then since A is real, $A = \bar{A}$ and $A\bar{X} = \bar{A}\bar{X} = \overline{AX} = \overline{\lambda X} = \bar{\lambda}\bar{X}$, so $\bar{X}$ is an eigenvector with eigenvalue $\bar{\lambda}$.

7. (a) $\|X - Y\| = \|X + Y\| \iff \langle X - Y, X - Y\rangle = \langle X + Y, X + Y\rangle \iff \langle X, X\rangle - \langle Y, X\rangle - \langle X, Y\rangle + \langle Y, Y\rangle = \langle X, X\rangle + \langle Y, X\rangle + \langle X, Y\rangle + \langle Y, Y\rangle \iff 0 = 2\langle Y, X\rangle + 2\langle X, Y\rangle = 4\langle X, Y\rangle \iff \langle X, Y\rangle = 0$.

(b) In $\mathbb{C}^2$ with the standard Hermitian dot product, let $X = (i, 0)$ and $Y = (1, 0)$. Then $\langle X, Y\rangle = i$ but $\|X - Y\| = \|X + Y\| = \sqrt{2}$. (Note: the proof in (a) no longer works since we no longer have $\langle X, Y\rangle = \langle Y, X\rangle$.)