Mathematics V1208y
Honors Mathematics IV

Answers to Final Examination

May 9, 2016

PART A: True/False. Decide whether the given statement is true or false, and give a brief reason for your answer (sketch of proof or counterexample). 4 points each.

1. True: if $A^T = -A$ and $B = A^{-1}$, then $I = (AB)^T = B^T A^T = -B^T A$, so $B^T = -A^{-1} = -B$.

2. False: try $\begin{pmatrix} 0 & 2 \\ 1 & 0 \end{pmatrix}$.

3. True: if $\sum a_i \mathbf{v}_i = 0$, then for each j, $0 = \mathbf{v}_j \cdot \sum a_i \mathbf{v}_i = \sum a_i \, \mathbf{v}_j \cdot \mathbf{v}_i = a_j \|\mathbf{v}_j\|^2$, but $\|\mathbf{v}_j\|^2 \neq 0$, so $a_j = 0$.

4. True: in fact its total derivative at every $\mathbf{v}$ is itself, for certainly

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{L(\mathbf{v}+\mathbf{h}) - L(\mathbf{v}) - L(\mathbf{h})}{\|\mathbf{h}\|} = \lim_{\mathbf{h}\to\mathbf{0}} \frac{\mathbf{0}}{\|\mathbf{h}\|} = \mathbf{0}.$$

5. False: C^1 implies differentiability, which implies continuity.

6. True: the vector field is closed, and the set is star-shaped.

PART B: Shorter proofs and computations. 7 points each.

7. Since H is Hermitian, it has real eigenvalues, and by the spectral theorem, H is diagonalizable. So for some A, $AHA^{-1} = D$ where D is diagonal with real entries. But $D - I = A(H - I)A^{-1}$ is also diagonal, and its diagonal entries $d_{ii} - 1$ are the eigenvalues of $H - I$. Since $d_{ii} - 1$ is imaginary and d_{ii} is real, $d_{ii} = 1$ for all i, so $D = I$, hence $H = I$.

8. For any $\mathbf{x} \in U$, take $\epsilon = |f(\mathbf{x})|$ in the definition of continuity; then there exists δ such that $\|\mathbf{y} - \mathbf{x}\| < \delta$ implies $|f(\mathbf{y}) - f(\mathbf{x})| < |f(\mathbf{x})|$, and hence $f(\mathbf{y}) \neq 0$, so that $\mathbf{y} \in U$. Therefore $B_\delta(\mathbf{x}) \subseteq U$.

9. Let $H(s,t) = (s^2 - t^2, s^2 + t^2, st)$; then $g = f \circ H$. By the chain rule, $D_g(s,t) = D_f(H(s,t))\, D_H(s,t)$, or

$$\left(\frac{\partial g}{\partial s} \quad \frac{\partial g}{\partial t}\right) = \left(\frac{\partial f}{\partial x} \quad \frac{\partial f}{\partial y} \quad \frac{\partial f}{\partial z}\right) \begin{pmatrix} 2s & -2t \\ 2s & 2t \\ t & s \end{pmatrix};$$

taking the right-hand entry yields $\partial g/\partial t = -2t\,\partial f/\partial x + 2t\,\partial f/\partial y + s\,\partial f/\partial z$.

10. Let C be a curve in $\mathbb{R}^n$, parametrized by a piecewise C^1 map $\gamma : [a,b] \to \mathbb{R}^n$, and let f be a scalar field on a subset of $\mathbb{R}^n$ containing $C = \gamma([a,b])$. The integral of f with respect to arclength is defined as

$$\int_C f \,\|d\gamma\| = \int_a^b f(\gamma(t)) \|\gamma'(t)\|\, dt$$

if the right-hand integral exists. (It is unchanged by a forward reparametrization of C.)

11. Let $G = (G_1, G_2, G_3)$. Then

$$\begin{aligned} \nabla \cdot (f^2 G) &= \sum_{i=1}^{3} \frac{\partial}{\partial x_i}(f^2 G_i) \\ &= \sum_{i=1}^{3} \left(\frac{\partial (f^2)}{\partial x_i}\, G_i + f^2\, \frac{\partial G_i}{\partial x_i}\right) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{i=1}^{3} \left(2f \frac{\partial f}{\partial x_i} G_i + f^2 \frac{\partial G_i}{\partial x_i} \right) \\
&= 2f\, \nabla f \cdot G + f^2 \nabla \cdot G.
\end{aligned}
$$

12. The divergence of F is the constant scalar field 2. Let V be the unit upper ball, $V = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 \le 1, z \ge 0\}$, so that the boundary of V is the union of S with the disc $D = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 \le 1, z = 0\}$. If D is parametrized in the obvious way by the unit disc in the (x, y)-plane, it has outward normal $(0, 0, 1)$, so on D, $F \cdot \mathbf{n} = 0$. Then by the divergence theorem, $\iint_S F \cdot d\mathbf{r}^2 = \iint_S F \cdot d\mathbf{r}^2 + \iint_D F \cdot d\mathbf{r}^2 = \iint_{\partial V} F \cdot d\mathbf{r}^2 = \iiint_V 2\, dx\, dy\, dz = 4/3\pi$, the last equality simply because a ball of radius r has volume $4/3\, \pi r^3$.

PART C: Longer proofs and computations. 10 points each.

13. By Fubini $k(t) = \int_0^{t^2} \int_0^1 f(x, y)\, dy\, dx$. Let $g(z) = \int_0^z \int_0^1 f(x, y)\, dy\, dx$, and let $h(t) = t^2$. Then $k = g \circ h$. Since $\int_0^1 f(x, y)\, dy$ is a continuous function of x, by the 1st fundamental theorem of calculus g is differentiable and $g'(z) = \int_0^1 f(z, y)\, dy$. Then by the (1-variable!) chain rule, k is differentiable with $k'(t) = 2t \int_0^1 f(t^2, y)\, dy$.

One could try to do this by differentiating under the integral without using Fubini, but this proves tricky as we need the integrand to be C^1. (If we assume that, then it can be done.)

14. Parametrize R by $s : [1, 5] \times [-1, 1] \to R$ where $s(u, v) = (u - v, u + v)$. Then $\det D_s(u, v) = 2$, so the transformation formula says

$$
\iint_R x^2 y\, dx\, dy = \int_{-1}^{1} \int_{1}^{5} 2(u^2 - v^2)\, du\, dv = 160
$$

if I'm not wrong.

15. (a) Straightforward: the answer is $2\mathbf{a}$.

(b) By part (a), using Stokes, this equals $\frac{1}{2} \iint_S \mathrm{curl} H \cdot d\mathbf{r}^2 = \frac{1}{2} \oint_C H \cdot d\gamma$, where C is the unit circle in the (x, y)-plane. Using your favorite parametrization, say $\gamma(t) = (\cos t, \sin t, 0)$, you find that $H(\gamma(t)) = (0, 0, \cos t)$, so $H(\gamma(t)) \cdot \gamma'(t) = 0$, and the line integral is 0.

(c) Moral: If the mouth of your fishnet is in a plane parallel to the motion of the fish, you won't catch any, no matter what shape the net is.

16. The curl of F is $\partial Q/\partial x - \partial P/\partial y = h'(x) + h'(y) \ge 0$. Green's theorem then says that $\oint_{C_r} F \cdot ds = \iint_{D_r} (h'(x) + h'(y))\, dx\, dy$, where D_r is the disc of radius r. For $r \ge 0$, let $f_r(x, y) = h'(x) + h'(y)$ if $x^2 + y^2 \le r^2$, 0 otherwise. Then $r' \le r$ implies $f_{r'} \le f_r$, so by comparison

$$
\begin{aligned}
\iint_{D'_r} (h'(x) + h'(y))\, dx\, dy &= \int_{-r}^{r} \int_{-r}^{r} f_{r'}(x, y)\, dx\, dy \\
&\le \int_{-r}^{r} \int_{-r}^{r} f_r(x, y)\, dx\, dy \\
&= \iint_{D_r} (h'(x) + h'(y))\, dx\, dy.
\end{aligned}
$$

17. (a) By the divergence theorem, the surface integral equals the integral of the divergence over the ball of radius r, which is now constant when $r \ge 1$, so $a = 0$.

(b) By Stokes's theorem, the surface integral over the northern hemisphere equals the line integral of G over the equator; the integral over the southern hemisphere equals the same integral, but with a backwards parametrization, so they cancel. Hence $a = b = 0$.