Mathematics V1208y
Honors Mathematics IV

Final Examination

May 9, 2016

If a region has graph type; or if a parametrization is counterclockwise; or if a reparametrization is forward or backward, just say so. You don't have to prove it. Good luck!

PART A: True/False. Decide whether the given statement is true or false, and give a brief reason for your answer (sketch of proof or counterexample). 4 points each.

1. If an invertible matrix A is skew-symmetric (i.e. $A_{ji} = -A_{ij}$), then so is its inverse.

2. The eigenvalues of a matrix with rational entries are rational.

3. If the nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$ are nonzero and all orthogonal, then they are linearly independent.

4. Any linear map $L : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable.

5. There exists a scalar field $f : \mathbb{R}^3 \to \mathbb{R}$ whose partial derivatives are all continuous, but which is not itself continuous.

6. If $U \subseteq \mathbb{R}^2$ is the union of the two rectangles $(1, 2) \times (-1, 3)$ and $(-1, 3) \times (1, 2)$, then the vector field $F(x, y) = (x/(x^2 + y^2),\ y/(x^2 + y^2))$ is a gradient on U.

PART B: Shorter proofs and computations. 7 points each.

7. If A is a Hermitian matrix and all the eigenvalues of $A - I$ are imaginary (that is, real multiples of i), prove that $A = I$.

8. Let $f : \mathbb{R}^n \to \mathbb{R}$ be any continuous scalar field. Show that $U = \{\mathbf{x} \in \mathbb{R}^n \mid f(\mathbf{x}) \neq 0\}$ is an open set.

9. Use the chain rule to express $\frac{\partial g}{\partial t}$ in terms of partial derivatives of $f(x, y, z)$ if $g(s, t) = f(s^2 - t^2, s^2 + t^2, st)$, and $f : \mathbb{R}^3 \to \mathbb{R}$ is a C^1 function.

10. Give a careful definition of the integral of a scalar field with respect to arclength.

11. Show that if f is a scalar field and G is a vector field on $\mathbb{R}^3$, then $\nabla \cdot (f^2 G) = 2f\, \nabla f \cdot G + f^2\, \nabla \cdot G$. Be sure you understand what each term means!

12. Using the divergence theorem, compute the surface integral $\int_S F \cdot d\mathbf{r}^2$, where S is the unit upper hemisphere in $\mathbb{R}^3$, and $F(x, y, z) = (x, y, 0)$.

PART C: Longer proofs and computations. 10 points each.

13. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is continuous, and let $k(t) = \int_0^1 \int_0^{t^2} f(x,y)\,dx\,dy$. Prove that k is differentiable and express its derivative in terms of a single integral. State clearly what theorems you are using. (Hint: recall we proved that $\int_0^1 f(x,y)\,dy$ is a continuous function of x.)

14. Let R be the rectangle in $\mathbb{R}^2$ with vertices at $(0,2)$, $(2,0)$, $(4,6)$, and $(6,4)$. Use the transformation formula to express $\iint_R xy\,dx\,dy$ as a double integral with constant limits of integration, and evaluate it.

15. Let $\mathbf{a} = (1,0,0) \in \mathbb{R}^3$, and let H be the vector field given by $H(\mathbf{x}) = \mathbf{a} \times \mathbf{x}$.

(a) Compute the curl of H.

(b) If D is the unit disk in the plane, and $r : D \to S$ is the parametric surface with $r(u,v) = (u, v, (1-u^2-v^2)\,e^{7u^2v}\cos v)$, compute $\iint_S \mathbf{a} \cdot d\mathbf{r}^2$.

(c) (Optional) Interpret in terms of fish.

16. Let g and $h : \mathbb{R} \to \mathbb{R}$ be increasing C^1 functions, and let F be the vector field on $\mathbb{R}^2$ given by $F(x,y) = (g(x) - h(y),\, h(x) - g(y))$. Use Green's theorem to show that, if C_r is the circle of radius r, then the (counterclockwise) line integral $\oint_{C_r} F \cdot d\gamma$ is an increasing function of r.

17. Let F be a C^1 vector field on $\mathbb{R}^3$. Let S_r be the sphere of radius r centered at $\mathbf{0}$, and let $\mathbf{n}$ be the unit outward normal. Suppose there exist constants a, b such that for all $r > 1$,

$$\iint_{S_r} F \cdot d\mathbf{r}^2 = ar + b.$$

(a) What conditions must a, b satisfy if $\operatorname{div} F(\mathbf{x}) = 0$ whenever $\|\mathbf{x}\| \geq 1$?

(b) What conditions must a, b satisfy if there exists a vector field G on $\mathbb{R}^3$ so that $\operatorname{curl} G(\mathbf{x}) = F(\mathbf{x})$ whenever $\|\mathbf{x}\| \geq 1$?