Mathematics V1208y

Honors Mathematics B

Answers to Midterm Exam

March 9, 2016

1. A matrix is in *reduced row-echelon form* if: (i) in each row, the first nonzero entry (if any) is 1, called a *leading 1*; (ii) each leading 1 is to the right of those above it; (iii) each leading 1 is the only nonzero entry in its column.

2. (a) The characteristic polynomial is $(\lambda+4)(\lambda-5)+18 = \lambda^2+\lambda-2 = (\lambda-2)(\lambda+1)$, so eigenvalues are 2 and -1. The null spaces of $A-2I$ and $A+I$ have bases $(1,1)$ and $(2,1)$ respectively (one can do this by Gauss-Jordan elimination on the respective matrices, or just write them down and eyeball it), so the change of basis from the basis of eigenvectors to the standard basis is $\left(\begin{smallmatrix}1&2\\1&1\end{smallmatrix}\right)$. We can invert this using $\left(\begin{smallmatrix}a&b\\c&d\end{smallmatrix}\right)^{-1} = \frac{1}{ad-bc}\left(\begin{smallmatrix}d&-b\\-c&a\end{smallmatrix}\right)$ to express A as $\left(\begin{smallmatrix}-4&6\\-3&5\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&2\\1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}2&0\\0&-1\end{smallmatrix}\right)\left(\begin{smallmatrix}-1&2\\1&-1\end{smallmatrix}\right)$.

(b) Multiply out $A^k = \left(\begin{smallmatrix}1&2\\1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}2&0\\0&-1\end{smallmatrix}\right)^k\left(\begin{smallmatrix}-1&2\\1&-1\end{smallmatrix}\right) = \left(\begin{smallmatrix}1&2\\1&1\end{smallmatrix}\right)\left(\begin{smallmatrix}2^k&0\\0&(-1)^k\end{smallmatrix}\right)\left(\begin{smallmatrix}-1&2\\1&-1\end{smallmatrix}\right)$ to obtain the upper right-hand entry $2^{k+1}+2(-1)^{k+1}$.

3. (a) Expand by minors along the last row to get $\det B = (-1)^{1+6}\, 9\, \det B^{61} = -9 \det I_5 = -9$.
(b) Since $\det(B^7) = (\det B)^7 = (-9)^7 \neq 0$ and B^7 is a 6×6 matrix, we must have $\operatorname{rank} B^7 = 6$.

4. If $n > 2$, then there are at least three rows, so we may subtract row 2 from row 3, then row 1 from row 2 (both elementary row operations of type III, which do not change the determinant) to obtain a matrix with two rows whose entries are all 1. This has determinant zero by axiom D3 of determinants.

On the other hand, if $n \leq 2$, then $\det() = 1$, $\det(2) = 2$, and $\det\left(\begin{smallmatrix}2&3\\3&4\end{smallmatrix}\right) = -1$.

5. (a) If $v \in \operatorname{im} T$, then there exists $w \in \mathbb{R}^n$ such that $v = T(w)$. Then $S(v) = S(T(w)) = S \circ T(w) = 0$, so $v \in \ker S$.

(b) By (a), $\dim \operatorname{im} T \leq \dim \ker S$, hence $\operatorname{rank} T \leq \operatorname{nullity} S$, hence $\operatorname{rank} T \leq n - \operatorname{rank} S$ by rank-nullity, hence $\operatorname{rank} S + \operatorname{rank} T \leq n$.

6. (a) If v is an eigenvector with eigenvalue λ, then $\lambda v = Dv = D^2 v = D\lambda v = \lambda D v = \lambda^2 v$. Since $v \neq 0$, $\lambda = \lambda^2$. Hence $(\lambda-1)\lambda = 0$, so $\lambda = 1$ or $\lambda = 0$.

(b) Let V_λ denote the λ-eigenspace of D. If $v \in V_1$, then $Dv = v$, so $v \in \operatorname{im} T_D$. Conversely, if $v \in \operatorname{im} T_D$, then $v = Dw$ for some w, but then $Dv = DDw = Dw = v$, so $v \in V_1$.

(c) Since $V_0 = \ker T_D$, $\dim V_0 = \operatorname{nullity} T_D$. But by (b), $\dim V_1 = \operatorname{rank} T_D$. By rank-nullity their dimensions sum to n. If $u_1, \ldots, u_k$ and $v_1, \ldots, v_{n-k}$ are bases for V_0 and V_1, respectively, then in fact $u_1, \ldots, u_k, v_1, \ldots, v_{n-k}$ must be linearly independent. For if $\sum_i a_i u_i + \sum_j b_j v_j = 0$, then applying T_D to both sides, we find $\sum_j b_j v_j = 0$, hence each $b_j = 0$ by independence of v_j, hence $\sum_i a_i u_i = 0$, hence each $a_i = 0$ by independence of u_i. As a sequence of length n which is independent, $u_1, \ldots, u_k, v_1, \ldots, v_{n-k}$ must be a basis for $\mathbb{R}^n$. But it consists of eigenvectors of D, so D is diagonalizable.

7. Proof 1: If E has n distinct eigenvalues, then the n associated eigenvectors are independent and hence form a basis. So E is diagonalizable, $E = BDB^{-1}$, where D is diagonal with diagonal entries $\lambda_1, \ldots, \lambda_n$. Then $\det E = \det B \det D \det B^{-1} = \det B\,(\lambda_1 \cdots \lambda_n)/\det B = \lambda_1 \cdots \lambda_n$.

Proof 2: If E has n distinct eigenvalues, then its characteristic polynomial is $\det(\lambda I - E) = \prod_{i=1}^n (\lambda - \lambda_i)$. Plug in $\lambda = 0$ to get $(-1)^n \det E = \det(-E) = \prod_{i=1}^n (-\lambda_i) = (-1)^n \lambda_1 \cdots \lambda_n$.