Mathematics V1208y
Honors Mathematics B

Assignment #7
Due March 25, 2016

Reading: Apostol §8 (pp. 243–281).

1. Apostol §1.17 (p. 30) 2, 3, 4.

***2.** Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by $(1,1,1,1)$, $(-1,4,4,-1)$, and $(4,-2,2,0)$.

3. Apostol §5.5 (pp. 118–120) 3, 4, 5, 6.

4. Apostol §5.11 (pp. 124–126) 1, 2, 6, 8, ***13**, 14.

Hint for 13: If $v, w \in \mathbb{R}^n$ are regarded as $n \times 1$ matrices, then $v \cdot w = v^t w$.

***5.** Let V be a finite-dimensional Euclidean or Hermitian space, $U \subset V$ any subspace. Show that $\dim U + \dim U^\perp = \dim V$.

***6.** (20 pts) Let U, V, W be finite-dimensional Euclidean or Hermitian spaces, $S : U \to V$ and $T : V \to W$ linear maps. Show that:

(a) $T^{**} = T$. (Choosing ONBs is perfectly legitimate, but working directly from the definition is in better taste.)

(b) $\ker T^* = (\operatorname{im} T)^\perp$.

(c) $\ker T = (\operatorname{im} T^*)^\perp$.

(d) $\operatorname{rank} T^* = \operatorname{rank} T$.

(e) Use the above to give an alternate proof that the row-rank of a real square matrix equals its column-rank.

(f) $(TS)^* = S^* T^*$.

(g) If T is invertible, then so is T^*, and $(T^*)^{-1} = (T^{-1})^*$.

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***7.** (20 pts) Prove the *finite-dimensional spectral theorem*:

Let V be a finite-dimensional Hermitian space and $T : V \to V$ a linear map. Then V has an orthonormal basis of T-eigenvectors if and only if T is normal (i.e. $TT^* = T^*T$).

Step 1. Prove the "only if" part by considering a diagonal matrix representation of T.

Step 2. Show that *any* linear map $T : V \to V$ can be written $T = H + iK$, where H and K are Hermitian.

Step 3. If T is normal, show that H and K commute in the above description.

Step 4. Let λ be an eigenvalue of H, and E_λ the corresponding eigenspace. Show that K maps E_λ into itself, and so by the Hermitian case of the spectral theorem, E_λ has an orthonormal basis of K-eigenvectors.

Step 5. Show that two commuting self-adjoint linear maps can be "simultaneously diagonalized," i.e. there exists an orthonormal basis of V consisting of eigenvectors for both H and K.

Step 6. Show V has an orthonormal basis of T-eigenvectors.

***8.** A linear map $T : V \to V$ on a finite-dimensional Hermitian space is called *non-negative* if it is self-adjoint and all its eigenvalues are positive or zero. Show, using the spectral theorem, that a non-negative linear map has an *unique* non-negative square root.