Mathematics V1208y
Honors Mathematics B

Assignment #6
Due March 5, 2016

Reading: Apostol §1.11–1.15 (pp. 14–28).

1. Apostol §1.13 (pp. 20–2) 3, 4, 5, 6, 9, 10, *13abd, *16ac. In 13b, you needn't prove that V is a linear space (which we already know), just that (x, y) is an inner product.

***2.** (a) If $A, B \in M_{n,n}$ *commute* (that is, $AB = BA$), and if $V_\lambda \subset \mathbb{R}^n$ is the λ-eigenspace of A, show that $T_B(V_\lambda) \subset V_\lambda$.

(b) If a real matrix $A \in M_{n,n}$ has n distinct real eigenvalues, describe all the real matrices $B \in M_{n,n}$ that commute with A and show that they are all diagonalizable.

3. Give an explicit formula, in terms of k, for the upper right-hand entry of A^k, where $A = \begin{pmatrix} 15 & -8 \\ 24 & -13 \end{pmatrix}$.

***4.** Let A be the matrix of rotation through an angle θ, namely $A = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$. Notice that, if you believe this is indeed rotation through θ, then applying it n times rotates through $n\theta$, so $A^n = \begin{pmatrix} \cos n\theta & \sin n\theta \\ -\sin n\theta & \cos n\theta \end{pmatrix}$.

(a) Prove that the eigenvalues of A are $\cos\theta \pm i \sin\theta$. Diagonalize A.

(b) Find formulas for $\cos 3\theta$ and $\sin 3\theta$ in terms of $\sin\theta$ and $\cos\theta$. Extra credit: do the same for $\cos n\theta$ and $\sin n\theta$. (Use the binomial theorem, or induction.)

***5.** (30 pts) If A is an $n \times n$ matrix with complex entries, a *square root* of A is, not surprisingly, an $n \times n$ matrix B such that $B^2 = A$.

(a) If a is any nonzero complex number, prove that it has exactly 2 complex square roots. (You may use the fundamental theorem of algebra.)

(b) If A is a diagonal matrix with nonzero entries on the diagonal, show that A has (at least) 2^n distinct square roots.

(c) If A is a diagonalizable matrix with nonzero eigenvalues, show that A has (at least) 2^n distinct square roots.

(d) Generalize Apostol §4.4 #4 (from the last assignment) to show that if B^2 has eigenvalue λ^2 for *any* complex λ, then B has eigenvalue either λ or $-\lambda$. Must the associated eigenvectors also be eigenvectors of B^2?

(e) If A is as in (c) and has n distinct eigenvalues, show that the 2^n square roots you constructed are the *only* square roots of A. Hint: use (d)!

(f) Find the square roots of the matrix $A = \begin{pmatrix} -11 & 15 \\ -20 & 24 \end{pmatrix}$.

(g) Show that the matrix $A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ has only 2 square roots. Hence the hypothesis that the eigenvalues be nonzero is necessary in (c).

(h) Show that the 2×2 identity matrix has at least 5 square roots. Hence the hypothesis that the eigenvalues be distinct is necessary in (e).

(i) Show that the 2×2 zero matrix has an infinite number of square roots.