Mathematics V1208y
Honors Mathematics B

Assignment #4
Due February 19, 2016

From now on, all assignments are from Volume II of Apostol.

Reading: Apostol §4.

1. Apostol §3.6 (pp. 79–81) 1, 2, 3.

2. Apostol §3.11 (pp. 85–86) 1, 2, 3, 4, 7.

***3.** Show that if A is any $n \times n$ matrix, then $\det(cA) = c^n \det A$.

***4.** For a square matrix A with transpose A^t, prove that $\det A = \det A^t$ by showing that the right-hand side satisfies the axioms of the determinant. (Don't follow the proof given in Apostol.)

***5.** A square matrix A is said to be *upper-triangular* if $A_{ij} = 0$ whenever $i > j$. Prove that if A is upper-triangular and invertible, then its inverse is upper-triangular. Give a 2×2 example.

***6.** Use expansion by minors to prove that the determinant of an upper-triangular matrix is the product of its diagonal entries. (Don't follow the proof given in Apostol.)

***7.** (a) For any square matrix A, show that $A + A^t$ is symmetric and $A - A^t$ is skew-symmetric.

(b) Use (a) to show that any square matrix can be expressed *uniquely* as a sum of one symmetric and one skew-symmetric matrix.

(c) Use a previous problem to show that if n is odd, then any $n \times n$ skew-symmetric matrix has determinant 0.

(d) Show that any nonzero 3×3 skew-symmetric matrix has rank exactly 2.
Hint: note that diagonal entries must vanish, and rule out all other possibilities.

***8.** Given fixed column vectors $Y_1, Y_2, \ldots, Y_{n-1} \in \mathbb{R}^n$, define a linear map $f : \mathbb{R}^n \to \mathbb{R}$ by

$$f(X) = \det(Y_1, Y_2, \ldots, Y_{n-1}, X),$$

that is, the determinant of the square matrix with columns Y_1 through Y_{n-1} and X. Describe the kernel of f in terms of the vectors $Y_1, \ldots, Y_{n-1}$, and prove that your description is correct.

CONTINUED OVERLEAF...

***9.** For $x_1, \ldots, x_n \in \mathbb{R}$ the *Vandermonde determinant* is defined to be

$$\det \begin{pmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{pmatrix}.$$

Prove that it equals $\prod_{1 \le i < j \le n} (x_j - x_i)$, the product of all differences $x_j - x_i$ with $i < j$. [Cf. problem 3a on p. 80 of Apostol.] Hint: subtract x_1 times the kth row from the $k+1$st for $k = n-1, n-2, n-3, \ldots, 1$. Then expand by minors and use induction.