Mathematics V1208y
Honors Mathematics B

Assignment #2
Due February 5, 2016

Buy Volume II of Apostol (second edition).

Read Apostol Vol. I, §16.13–16.19 (pp. 597–613) or, equivalently, Apostol Vol. II, §2.13–2.19 (pp. 51–67).

In Apostol, do the following problems. Most are unstarred and for quick perusal. Notes on terminology: what Apostol calls *null space* is what we call *kernel*; what he (confusingly) calls *range* is what we call *image*; what he calls V_n is what we call $\mathbb{R}^n$.

Vol. I §16.4 (pp. 582–3) = Vol. II §2.8 (pp. 42–44) 4, 5, 6, 7, 8, 9, 10, 17, 18, 24, 28.

Vol. I §16.8 (pp. 589–90) = Vol. II §2.8 (pp. 42–44) 4, 6, 10, 22, 23, 24, 25, 27.

Vol. I §16.12 (pp. 596–7) = Vol. II §2.12 (pp. 50–51) 1, *2, *5, 6, 11, 12, 13.

Vol. I §16.16 (pp. 603–4) = Vol. II §2.16 (pp. 57–58) 1, 2, 10, 14.

Also do the following. Starred problems are worth 10 points each. General hints: It will frequently be useful (a) to choose a basis for a subspace and extend to a basis for the whole space; (b) to define a linear map by using the construction principle.

***1.** Prove the space $\mathcal{F}(\mathbb{N}, \mathbb{R})$ of sequences is infinite-dimensional.

2. Show that if two finite-dimensional vector spaces are isomorphic, then they have the same dimension.

***3.** If W is a finite-dimensional vector space and $V \subset W$ is a subspace, prove that: (a) V is finite-dimensional; (b) $\dim V \leq \dim W$; (c) if $\dim V = \dim W$, then $V = W$.

***4.** (a) In $\mathbb{R}^3$, a *line* through the origin is the linear subspace $L = \{t\mathbf{c} \,|\, t \in \mathbb{R}\}$ for some constant $\mathbf{c} \neq \mathbf{0} \in \mathbb{R}^3$. Find a basis for L. Show that a line is 1-dimensional (!).

(b) In $\mathbb{R}^3$, a *plane* through the origin is the linear subspace $P = \{\mathbf{x} \in \mathbb{R}^3 \,|\, \mathbf{x} \cdot \mathbf{c} = \mathbf{0}\}$ for some constant $\mathbf{c} \neq \mathbf{0} \in \mathbb{R}^3$. Find a basis for P in terms of the components c_1, c_2, c_3 of $\mathbf{c}$. Show that a plane is 2-dimensional (!!). Note: watch out, any c_i could be 0.

(c) Prove that no line through the origin contains a plane through the origin.

5. Let V be a subspace of a finite-dimensional vector space W. Following the lecture, choose a basis $X_1, \ldots, X_k$ for V and extend to a basis $X_1, \ldots, X_n$ for W. Given $X = \sum_{i=1}^{n} a_i X_i \in W$, show that $X \in V \Leftrightarrow a_{k+1} = a_{k+2} = \cdots = a_n = 0$.

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6. Let U_n be the vector space of polynomial functions $\mathbb{R} \to \mathbb{R}$ of degree $\leq n$.

(a) Show that the map $G : U_n \to \mathbb{R}^k$ given by $G(f) = (f(1), f(2), \ldots, f(k))$ is linear, and is surjective when $k \leq n+1$. Hint: consider the polynomials $f_i(x) = (x-1)(x-2)(x-3)\cdots(x-i+1)(x-i-1)\cdots(x-k)$.

(b) Use rank-nullity to determine the dimension of the subspace of U_n consisting of polynomials satisfying $f(1) = f(2) = \cdots = f(k) = 0$.

***7.** Let U, V be finite-dimensional subspaces of a vector space X, and let

$$W = \{au + bv \,|\, u \in U,\ v \in V,\ a, b \in \mathbb{R}\}.$$

(a) Show that W is a subspace of X.

(b) Show there exist bases $u_1, \ldots, u_m$ and $v_1, \ldots, v_n$ for U and V, respectively, which both begin with the same basis for $U \cap V$, namely $u_1 = v_1, \ldots, u_k = v_k$.

(c) Show that $u_1, \ldots, u_m, v_{k+1}, \ldots, v_n$ is then a basis for W.

(d) Prove that $\dim W = \dim U + \dim V - \dim U \cap V$.

(e) Illustrate with an example in $X = \mathbb{R}^3$.

***8.** Let V be a vector space of dimension n.

(a) If $\mathbf{v} \neq \mathbf{0} \in V$, show there exists a linear $T : V \to \mathbb{R}$ such that $T(\mathbf{v}) = 1$.

(b) If $W \subset V$ is a *hyperplane*, that is, a subspace of dimension $n-1$, show there exists a linear $T : V \to \mathbb{R}$ such that $W = \ker T$.

(c) More generally, if $W \subset V$ is any subspace, show that for some k there exists a linear $T : V \to \mathbb{R}^k$ such that $W = \ker T$.

9. Let V be a finite-dimensional vector space, and let $S \subset V$ be an *affine hyperplane*, that is,

$$S = \{u + x_0 \,|\, u \in U\}$$

where $U \subseteq V$ is a hyperplane and x_0 is some fixed vector in V but not in U. Show that there is an *unique* linear $T : V \to \mathbb{R}$ such that $T(v) = 1 \Leftrightarrow v \in S$.

10. Let V and W be finite-dimensional vector spaces. Show that a linear map $T : V \to W$ has a right inverse if and only if it is surjective, and a left inverse if and only if it is injective.

***11.** Let U and V be vector spaces of dimensions m and n, respectively.

(a) If there exists a linear surjection $T : U \to V$, prove that $m \geq n$.

(b) If there exists a linear injection $T : U \to V$, prove that $m \leq n$.

***12.** (a) Let U and V be vector spaces of dimensions m and n, respectively. Suppose that $T : U \to V$ and $S : V \to U$ are linear maps such that $T \circ S = \mathrm{id}_V$. Prove that $m \geq n$. Hint: use A2#4 from last semester.

(b) Suppose $A \in M_{n\times m}$ and $B \in M_{m\times n}$ are matrices such that $AB = I_n$, the $n \times n$ identity matrix of A1#2. Prove that $m \geq n$. (Note: this is a concrete statement about matrices that would be very difficult to prove without the abstract theory.)