Mathematics V1208y
Honors Mathematics IV

Assignment #12
Due April 29, 2016

Reading: Apostol, §§12.14–12.20 (pp. 443–462).

***1.** Let $f : [a, b] \to \mathbb{R}$ be a differentiable function. Express the subset of $\mathbb{R}^3$ swept out by the graph of f as it rotates about the horizontal x-axis as a parametric surface in $\mathbb{R}^3$: the *surface of revolution*. Write down an expression for the surface area between $x = a$ and $x = b$, as an integral from a to b.

2. The *torus* $T \subseteq \mathbb{R}^3$ is the set of all vectors of the form $3\mathbf{x} + \mathbf{y}$, where $\mathbf{x}$ is of unit length and orthogonal to $\mathbf{e}_1$, and $\mathbf{y}$ is of unit length and orthogonal to $\mathbf{x} \times \mathbf{e}_1$. Express the torus as a piecewise parametric smooth surface. Integrate your favorite function or vector field.

***3.** Let $r : T \to S$ be a parametric surface and $\gamma : [a, b] \to T$ a path in the plane, so that $r \circ \gamma : [a, b] \to S$ is a path in space. Show that the tangent vector to $r \circ \gamma$ is orthogonal to the outward normal.

4. Suppose that a piecewise parametric surface can be expressed *implicitly*, that is, $S = S_1 \cup \cdots \cup S_k$ with S_i parametrized by $r_i : T_i \to S_i$ can be expressed as

$$S = \{\mathbf{x} \in \mathbb{R}^3 \mid f(\mathbf{x}) = 0\}$$

for a differentiable $f : \mathbb{R}^3 \to \mathbb{R}$. Show that at each point of S, ∇f is a nonzero scalar multiple of the outward normal $\partial r_i/\partial u \times \partial r_i/\partial v$. (Hint: what is $f \circ r_i$?) Give an example.

5. Apostol §12.4 (p. 424) 8, 10.

6. Apostol §12.6 (pp. 429–30) 2, *4.

7. Apostol §12.10 (pp. 436–8) *5, *6, *7.

Before doing the above problems, see the note on the back of this sheet about Apostol's notation for line and surface integrals.

8. Apostol §12.13 (pp. 442–3) *1, 3, *6, *11.

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A NOTE ON APOSTOL'S NOTATION FOR INTEGRALS

Before attempting problems where Apostol gives surface integrals, be sure to read his §12.9 (pp. 434–6), "Other notations for surface integrals." I have summarized 3 key points below.

(1) **Line integrals**. The LHS and RHS are Apostol's notation and my notation, respectively, for line integrals of vector fields in $\mathbb{R}^3$:

$$\int_C P\,dx + Q\,dy + R\,dz = \int_C (P, Q, R) \cdot d\gamma.$$

In some sense we agree, since formally $d\gamma = \gamma'(t)\,dt = (\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt})\,dt = (dx, dy, dt)$. Just don't ask what these quantities really are! It's just formal notation. The real definition, as you know, is an integral from a to b with respect to t.

(2) **Surface integrals of scalar fields**, otherwise known as integrals with respect to surface area. Once again, Apostol's notation and my notation are:

$$\iint_S \phi\,dS = \iint_S \phi\,\|d\mathbf{r}^2\|.$$

Again, the idea is that formally $dS = \|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\|\,du\,dv$, and the factors of du and dv "cancel".

(3) **Surface integrals of vector fields.** Here our two notations are

$$\iint_S F \cdot \mathbf{n}\,dS = \iint_S F \cdot d\mathbf{r}^2.$$

Yet again, formally this sort of makes sense since

$$\mathbf{n} = \frac{\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}}{\|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\|}, \quad dS = \|\frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\|\,du\,dv, \quad d\mathbf{r}^2 = \frac{\partial \mathbf{r}}{\partial u} \times \frac{\partial \mathbf{r}}{\partial v}\,du\,dv.$$

When $F = (P, Q, R)$, Apostol sometimes denotes the same surface integral by

$$\iint_S P\,dy \wedge dz + Q\,dz \wedge dx + R\,dx \wedge dy.$$

The formal justification here is that the first component of $d\mathbf{r}^2$ is

$$\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v} - \frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)\,du\,dv = \frac{\partial(y, z)}{\partial(u, v)}\,du\,dv,$$

using the notation for the Jacobian determinant introduced on p. 394, and similarly for the other two components.