Mathematics V1208y
Honors Mathematics B

Assignment #1
Due January 29, 2016

Carefully read the course syllabus.

Read Apostol vol. I, §15.1–15.8 (pp. 551–560) and §16.1–16.10 (pp. 578–594), or, equivalently, Apostol vol. II, §1.1–1.9 (pp. 3–13) and §2.1–2.10 (pp. 31–48).

In general, only the starred problems are to be written up and handed in. The others are supplementary problems for your own benefit. But this week there are no unstarred problems!

In each of the exercises below, V denotes a vector space.

***1.** Let $v_1, v_2, \ldots$ be an infinite list or sequence of elements of V (that is, a function from $\mathbb{N}$ to V).

(a) Prove that if $v_1, \ldots, v_k$ spans V, then so does $v_1, \ldots, v_n$ for any $n \geq k$.

(b) Prove that if $v_1, \ldots, v_n$ is linearly independent, then so is $v_1, \ldots, v_k$ for any $k \leq n$.

***2.** The identity map $\mathrm{id} : \mathbb{R}^n \to \mathbb{R}^n$ is linear. By the theorem proved last semester, it must equal T_I for some $n \times n$ matrix I. State a formula for the (i, j)th entry $I_{i,j}$ of I and prove that your formula is correct.

***3.** Let $T : V \to V$ be a linear map. Define $T^n : V \to V$ recursively by $T^0 = \mathrm{id} : V \to V$ and $T^{n+1} = T^n \circ T$. That is, $T^n = T \circ \cdots \circ T$ (n times).

(a) If $T^3 = 0$, prove that $\mathrm{id} - T$ is an isomorphism. Hint: consider $\mathrm{id} + T + T^2$.

(b) Show that $T_A : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies the requirements of (a) when

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

(c) Generalize to the case $T^n = 0$ for some n.

(d) Discuss the relationship to the geometric series.

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***4.** Let $v_1, \ldots, v_n \in V$. If $W \subset V$ is a subspace containing $v_1, \ldots, v_n$, prove that the linear span $L(v_1, \ldots, v_n) \subset W$. (We express this informally by saying that $L(v_1, \ldots, v_n)$ is the smallest subspace containing the v_i.)

***5.** Let $n \in \mathbb{N}$. In $\mathcal{F}(\mathbb{R}, \mathbb{R})$, the space of functions $\mathbb{R} \to \mathbb{R}$, prove that $\sin x$, $\sin 2x$, $\sin 4x$, $\sin 8x, \ldots, \sin 2^n x$ is linearly independent.

***6.** Let $n \in \mathbb{N}$. In $\mathcal{F}(\mathbb{R}, \mathbb{R})$, the space of functions $\mathbb{R} \to \mathbb{R}$, prove that 1, $1 + x$, $1 + x + x^2$, $\ldots, 1 + x + x^2 + \cdots + x^n$ is linearly independent.

***7.** Let $v_1, \ldots, v_n \in V$.

(a) If $v_n \in L(v_1, \ldots, v_{n-1})$, prove that $v_1, \ldots, v_n$ is linearly dependent.

(b) More generally, prove that if any v_k can be expressed as a linear combination of $v_1, \ldots, v_{k-1}$, then $v_1, \ldots, v_n$ is linearly dependent. (This is the converse of the "lemma of the preceding elements.")

***8.** Give a proof or counterexample to each of the following, where A and B are matrices of size $m \times n$ and $n \times p$, respectively.

(a) If some row of A has all entries 0, then the same is true of AB.

(b) If some column of A has all entries 0, then the same is true of AB.

(c) If some column of B has all entries 0, then the same is true of AB.

(d) If two columns of B are identical, then the same is true of AB.

***9.** (a) Find a basis for the subspace W of $\mathbb{R}^n$ defined by

$$W = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid \sum_{i=1}^{n} x_i = 0\}.$$

What is the dimension of W?

(b) Sketch W in the cases $n = 2$ and $n = 3$.

***10.** Let V and W be vector spaces. On the Cartesian product $V \times W$, define addition by $(v, w) + (v', w') = (v + v', w + w')$ and scalar multiplication by $c(v, w) = (cv, cw)$.

(a) Prove that with these operations, $V \times W$ is a vector space. (We call it the *direct sum* of V and W and denote it $V \oplus W$.)

(b) Prove that if V and W are finite-dimensional, then so is $V \oplus W$. State and prove a formula for $\dim V \oplus W$ in terms of $\dim V$ and $\dim W$.