Mathematics G4403y
Modern Geometry

Answers to Practice Final

May 12, 2014

1. The action of $O(n)$ on $\mathbf{R}^n$ preserves g and interchanges all radial lines, so it suffices to consider the line through e_n. If $\gamma(t)$ is parametrized by arclength, then $\langle \gamma'(t), \gamma'(t)\rangle = 1$ and hence $0 = d\langle \gamma', \gamma'\rangle = 2\langle D\gamma'/dt, \gamma'\rangle$. So $D\gamma'/dt \perp \gamma'$ (wrt g, hence wrt the standard metric too), so $D\gamma'/dt$ is tangent to a sphere S centered at 0. However, the action of $O(n-1) \subset O(n)$ preserves γ and g and hence also $D\gamma'/dt$ but acts on $T_{\gamma(t)}S$ fixing only 0. Hence $D\gamma'/dt = 0$.

2. By definition of normal coordinates, all radial lines are geodesics, so for all indices i, j and for all $a, b \in \mathbf{R}$, $0 = \nabla_{a\partial_i + b\partial_j}(a\partial_i + b\partial_j)$. The coefficient of ab is $\nabla_{\partial_i}\partial_j + \nabla_{\partial_j}\partial_i = 2\nabla_{\partial_i}\partial_j = 2\sum \Gamma_{ij}^k \partial_k$ since the Levi-Civita connection is torsion-free. Hence all $\Gamma_{ij}^k = 0$.

3. Choose a bi-invariant metric. We proved in Assignment 10 #7d that the exponential map in the Lie-theoretic sense agrees with the Riemannian exponential map $\exp_e$ based at $e \in G$. But a compact manifold is complete, and by the Hopf-Rinow theorem, on a complete manifold the exponential map based at any point is surjective.

4. (a) $II(x)$ is the curvature at 0 of the graph of $f(t) = (tx)^T A(tx) = t^2 x^T Ax$, namely $2x^T Ax$ (times the unit normal e_{n+1}). (b) The sectional curvature is the Gaussian curvature of the intersection of $\langle e_i, e_j, e_{n+1}\rangle$ with M, namely $\det II|_{\langle e_i, e_j\rangle} = 4(a_{ii}a_{jj} - a_{ij}^2)$. (c) Finally, by Gauss's formula, $R_{i,j,k,l} = II(i,l)II(j,k) - II(i,k)II(j,l) = 4(a_{il}a_{jk} - a_{ik}a_{jl})$. Notice this agrees with (b) when we take $k = j$, $l = i$.

5. For $u \in T_pM$, $v \in T_qN$, choose geodesics $\beta(s)$, $\gamma(t)$ through u, v; then $\beta \times \gamma : (-\epsilon, \epsilon) \times (-\delta, \delta) \to M \times N$ is an isometric immersion, so the sectional curvature of $\langle u, v\rangle$ is zero.

6. For any $p \in M$, let $v \in T_pM$ be the tangent vector along the line. Then $II(v) = 0$, so II is indefinite, so its two eigenvalues κ_1, κ_2 cannot both be nonzero of the same sign, so $\kappa_1\kappa_2 \leq 0$.

7. For $n < 2$ this is vacuous. For $n \geq 2$, since $TT^n = \mathbf{R}^n \times T^n$, the space of all 2-planes in TM is $\mathrm{Gr}(2, n) \times T^n$, which is compact. Hence the sectional curvature, if positive, would be bounded below by a positive constant. This contradicts Bonnet's theorem, since $\pi_1 T^n = \mathbf{Z}^n$.

8. (a) Both ∇_v and A_v are C^∞-linear in v, so $\nabla + A$ is too. And $(\nabla + A)(fs) = \nabla(fs) + A(fs) = (df)s + f\nabla s + fAs = (df)s + (\nabla + A)s$.

(b) $(\nabla + A)(\nabla + A)s = \nabla^2 s + A\nabla s + \nabla(As) = \nabla^2 s + A\nabla s + (dA)s - A\nabla s = \nabla^2 s + (dA)s$, so $F_{\nabla + A} = F_\nabla + dA$.

(c) If L has a flat connection ∇, then $c_1(L) = [F_\nabla] = [0] = 0$. Conversely, if $0 = c_1(L) = [F_\nabla] \in H^2(M, \mathbf{C})$, then $F_\nabla = dA$ for some $A \in \Omega^1(M, \mathbf{C})$. Let $\tilde{\nabla} = \nabla - A$; then by (b), $\tilde{\nabla}$ is flat.