Mathematics G4403y
Modern Geometry

Practice Final Exam
May 12, 2014

Attempt all eight problems. On each page of your blue book, write the number of the problem you work on inside a *circle*. You may use any results from lectures or homework, but be sure to refer to them clearly and state the source. In grading the exam, I will emphasize accuracy, brevity, and clarity. Good luck!

1. Let $M = \mathbf{R}^n$ equipped with the metric $g = f(r^2)(dx_1^2 + \cdots + dx_n^2)$ for $r^2 = x_1^2 + \cdots + x_n^2$ and $f : \mathbf{R} \to \mathbf{R}$ a given smooth positive function. Show that the lines through the origin, parametrized by arclength, are geodesics.

2. Show that in normal coordinates centered at p, all Christoffel symbols $\Gamma_{ij}^k(p) = 0$.

3. Show that the exponential map (as defined last semester) on a compact connected Lie group is surjective.

4. Let A be a symmetric $n \times n$ matrix, let $F : \mathbf{R}^n \to \mathbf{R}$ be $F(x) = x^T A x$, and let M be the graph of F in $\mathbf{R}^{n+1}$. At the origin, compute (a) the second fundamental form, (b) the sectional curvature of $\langle e_i, e_j \rangle$, and (c) the Riemann curvature tensor.

5. Let M, N be Riemannian manifolds and let $M \times N$ have the product metric. Show that any plane in $T_{p,q}(M \times N)$ spanned by one vector tangent to M and another tangent to N has sectional curvature zero.

6. An embedded surface in $\mathbf{R}^3$ is said to be *ruled* if it contains a straight line through every point. Show that such a surface (with the induced metric) has everywhere nonpositive Gaussian curvature.

7. Let $T^n = (S^1)^n$. Show that T^n has no metric of positive sectional curvature.

8. Let $L \to M$ be a complex line bundle with complex connection ∇, and let $A \in \Omega^1(M, \mathbf{C})$.

(a) Show that $\nabla + A$ is another connection on L.

(b) State and prove a formula for $F_{\nabla + A}$ in terms of F_{∇}.

(c) Show that L has a flat connection if and only if the first Chern class $c_1(L) = 0$.