Mathematics G4402x
Modern Geometry

Answers to Practice Final

1. For any $p, q \in M$, choose a smooth path $\gamma : [0,1] \to M$ with $\gamma(0) = p, \gamma(1) = q$. Then for any $g \in C^\infty(N)$, by the chain rule $(g \circ f \circ \gamma)'(t) = D_{f(\gamma(t))}g\, D_{\gamma(t)}f\, D_t\gamma = 0$, so $g \circ f \circ \gamma$ is constant and $g(f(p)) = g(f(q))$. Since there is a smooth function g separating any two distinct points, $f(p) = f(q)$; hence f is constant.

2. If $f : G \to H$ is as stated, then for any $g \in G$ the diagram

$$\begin{array}{ccc} G & \xrightarrow{\cdot g} & G \\ {\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ H & \xrightarrow{\cdot f(g)} & H \end{array}$$

commutes, and hence so does the derivative diagram

$$\begin{array}{ccc} T_eG & \xrightarrow{D_e g} & T_gG \\ {\scriptstyle D_e f}\big\downarrow & & \big\downarrow{\scriptstyle D_g f} \\ T_eH & \xrightarrow{D_e f(g)} & T_{f(g)}H. \end{array}$$

Since the rows are isomorphisms, the columns have the same rank, so rank $D_g f$ is independent of g. This must equal $\dim G$, for if not, the Rank Theorem says that f is not injective even in any neighborhood of e. Hence f is an immersion.

3. We compute in local coordinates x_i; let $\partial_i = \partial/\partial x_i$, $\eta = \sum_i \eta_i dx_i$, $X = \sum_i X_i \partial_i$, $Y = \sum_i Y_i \partial_i$. Then $d\eta = \sum_{i,j} (\partial_j \eta_i) dx_j \wedge dx_i$, so $d\eta(X,Y) = \sum_{i,j} (\partial_j \eta_i)(X_j Y_i - Y_j X_i)$. On the other hand, $X\eta(Y) = X(\sum_i \eta_i Y_i) = \sum_{i,j} X_j \partial_j (\eta_i Y_i) = \sum_{i,j} X_j (Y_i \partial_j \eta_i + \eta_i \partial_j Y_i)$, and similarly $Y\eta(X) = \sum_{i,j} Y_j (X_i \partial_j \eta_i + \eta_i \partial_j X_i)$. And $[X,Y] = \sum_{i,j} X_i (\partial_i Y_j) \partial_j - Y_j (\partial_j X_i) \partial_i = \sum_{i,j} (X_j \partial_j Y_i - Y_j \partial_j X_i) \partial_i$, so $\eta[X,Y] = \sum_{i,j} \eta_i (X_j \partial_j Y_i - Y_j \partial_j X_i)$. So the RHS becomes a sum over i, j of 6 terms, 4 of which cancel, leaving only the LHS.

4. Since $\eta(Y) \in \Omega^0(M)$, $L_X(\eta(Y)) = X\eta(Y)$. Also, we know $L_X Y = [X,Y]$. By the Cartan formula, $(L_X \eta)(Y) = (i_X d\eta + d i_X \eta)(Y) = d\eta(X,Y) + d(\eta(X))(Y) = d\eta(X,Y) + Y\eta(X)$. The result now follows from the previous problem.

5. Trivial if $k = 0$. If $k > 0$, then $M \times N$ compact, connected, and orientable implies that M and N are too. (To see orientability, observe that if ω is an orientation form for $M \times N$ and $v_1, \ldots, v_m$ are a basis for T_pM, then $i_{v_1} \cdots i_{v_m} \omega|_{\{p\} \times N}$ is an orientation form for N.) Hence $H^n(N) \cong \mathbb{R} \cong H^0(M)$, so by the Künneth formula, $H^n(M \times N) \neq 0$. If $0 < n < k$, this disagrees with the de Rham cohomology of S^k. So $n = 0$ or k. In the former case, N is connected of dimension 0, hence a point; in the latter case, the same is true of M.

6. Since d commutes with pullback, this realizes the de Rham complex of $\mathbb{RP}^n$ as a subcomplex of that of S^n. Since $\mathbb{RP}^n$ is connected, $H^0(\mathbb{RP}^n) = \mathbb{R}$. Since it is of dimension n, $H^k(\mathbb{RP}^n) = 0$ for $k > n$. Since $H^k(S^n) = 0$ for $0 < k < n$, every closed form ν of those degrees in this subcomplex is exact, $\nu = d\eta$. But then, if $\gamma : S^k \to S^k$ is the antipodal map, $(\eta + \gamma^*\eta)/2$ is in the subcomplex and $d(\eta + \gamma^*\eta)/2 = \nu$. Hence $H^k(\mathbb{RP}^n) = 0$ for $0 < k < n$. Finally, we know that for $\nu \in \Omega^n(S^n)$, $\nu = d\eta$ if and only if $\int_{S^n} \nu = 0$. If such a ν is also in the subcomplex, then $\nu = d(\eta + \gamma^*\eta)/2$ is exact in the subcomplex too, so $\int_{S^n}$ gives a well-defined injection $H^n(\mathbb{RP}^n) \to \mathbb{R}$. But γ preserves the orientation of S^n if and only if n is odd. (To see this, consider its action on the orientation form $\omega = i_{\mathbf{n}} dx_1 \wedge \cdots \wedge dx_{n+1}$.) So if n is even, any γ-invariant form has integral 0, so the injection is 0 and $H^n(\mathbb{RP}^n) = 0$. On the other hand, if n is odd, then ω is a γ-invariant form with nonzero integral, so the injection is nonzero and $H^n(\mathbb{RP}^n) \cong \mathbb{R}$.

7. Let $i : \mathbb{CP}^{n-1} \to V$ be the obvious inclusion, $\pi : V \to \mathbb{CP}^{n-1}$ the obvious projection. Then $\pi \circ i = \text{id}$, and $\Phi(t, [z_0, \ldots, z_n]) = [tz_0, z_1, \ldots, z_n]$ is a smooth homotopy from $i \circ \pi$ to id, so i induces an isomorphism on de Rham cohomology.

In the same way, include S^{2n-1} as the unit sphere in $U \cap V \simeq \mathbb{R}^{2n}\backslash 0$ and use the homotopy $\Phi(t, (x_1, \ldots, x_{2n})) = (x_1, \ldots, x_{2n})/(t + (1-t)||(x_1, \ldots, x_{2n})||)$ to get the desired isomorphism.

Now apply Mayer-Vietoris to get

$$\begin{array}{ccccccc} 0 & \longrightarrow & H^0\mathbb{CP}^n & \longrightarrow & H^0\mathbb{C}^n \oplus H^0\mathbb{CP}^{n-1} & \longrightarrow & H^0S^{2n-1} \\ & \longrightarrow & H^1\mathbb{CP}^n & \longrightarrow & H^1\mathbb{C}^n \oplus H^1\mathbb{CP}^{n-1} & \longrightarrow & H^1S^{2n-1} \\ & \longrightarrow & H^2\mathbb{CP}^n & \longrightarrow & H^2\mathbb{C}^n \oplus H^2\mathbb{CP}^{n-1} & \longrightarrow & H^2S^{2n-1} \\ & \longrightarrow & H^3\mathbb{CP}^n & \longrightarrow & H^3\mathbb{C}^n \oplus H^3\mathbb{CP}^{n-1} & \longrightarrow & \ldots \end{array}$$

The first row is just constants, so the first connecting homomorphism is zero. By induction we know $H^1\mathbb{CP}^{n-1} = 0$, so $H^1\mathbb{CP}^n = 0$. Likewise since $H^{k-1}S^{2n-1} = H^kS^{2n-1} = 0$ for $1 < k < 2n-1$, we find $H^k\mathbb{CP}^n \cong H^k\mathbb{CP}^{n-1}$ for $1 < k < 2n-1$. Finally we have

$$\begin{array}{ccccccc} & & & & & & H^{2n-2}S^{2n-1} \\ \longrightarrow & H^{2n-1}\mathbb{CP}^n & \longrightarrow & H^{2n-1}\mathbb{CP}^{n-1} & \longrightarrow & H^{2n-1}S^{2n-1} \\ \longrightarrow & H^{2n}\mathbb{CP}^n & \longrightarrow & H^{2n}\mathbb{CP}^{n-1}; & & \end{array}$$

the 1st, 3rd and 6th terms are 0 while the 4th is $\cong \mathbb{R}$, so we get $H^{2n-1}\mathbb{CP}^n = 0$, $H^{2n}\mathbb{CP}^n \cong \mathbb{R}$, as desired.