Mathematics G4402x
Modern Geometry

Practice Final Exam

Attempt all seven problems. You may use any results from lectures or assignments, but be sure to refer to them clearly and state the source.

In what follows, M and N always denote smooth manifolds.

1. If M is connected, $f : M \to N$ is smooth, and $D_p f = 0$ for all $p \in M$, then f is constant.

2. A smooth injective Lie group homomorphism is an immersion.

3. If $\eta \in \Omega^1(M)$ and $X, Y \in VF(M)$, then $d\eta(X, Y) = X\eta(Y) - Y\eta(X) - \eta[X, Y]$.

4. If $\eta \in \Omega^1(M)$ and $X, Y \in VF(M)$, then $L_X(\eta(Y)) = (L_X\eta)(Y) + \eta(L_X Y)$.

5. If S^k is diffeomorphic to a product $M \times N$, then either M or N is a point.

6. You showed in Assignment #6 that when a finite group Γ acts smoothly and freely on M, the pullback induces an isomorphism

$$\Omega^k(M/\Gamma) \cong \{\nu \in \Omega^k(M) \,|\, \gamma^*\nu = \nu \text{ for all } \gamma \in \Gamma\}.$$

Use this to show that $\dim H^k(\mathbb{RP}^n) = 1$ if $k = 0$ or if $k = n$ is odd, 0 otherwise.

7. In $\mathbb{CP}^n$, let $U = \{[x_0, \ldots, x_n] \,|\, x_0 \neq 0\}$ and $V = \mathbb{CP}^n \setminus [1, 0, \ldots, 0]$. Use homotopies to show that V has the same de Rham cohomology as $\mathbb{CP}^{n-1}$ and $U \cap V$ has the same de Rham cohomology as S^{2n-1}. Use this to show that $\dim H^k(\mathbb{CP}^n) = 1$ if k is even and $0 \leq k \leq 2n$, 0 otherwise.