Attempt all seven problems. You may use any results from lectures or assignments, but be sure to refer to them clearly and state the source.

In what follows, $M$ and $N$ always denote smooth manifolds.

1. If $M$ is connected, $f : M \to N$ is smooth, and $D_p f = 0$ for all $p \in M$, then $f$ is constant.

2. A smooth injective Lie group homomorphism is an immersion.

3. If $\eta \in \Omega^1(M)$ and $X,Y \in \mathcal{VF}(M)$, then $d\eta(X,Y) = X\eta(Y) - Y\eta(X) - \eta[X,Y]$.

4. If $\eta \in \Omega^1(M)$ and $X,Y \in \mathcal{VF}(M)$, then $L_X(\eta(Y)) = (L_X\eta)(Y) + \eta(L_X Y)$.

5. If $S^k$ is diffeomorphic to a product $M \times N$, then either $M$ or $N$ is a point.

6. You showed in Assignment #6 that when a finite group $\Gamma$ acts smoothly and freely on $M$, the pullback induces an isomorphism
   \[ \Omega^k(M/\Gamma) \cong \{ \nu \in \Omega^k(M) \mid \gamma^*\nu = \nu \text{ for all } \gamma \in \Gamma \} \]
   Use this to show that $\dim H^k(\mathbb{R}P^n) = 1$ if $k = 0$ or if $k = n$ is odd, 0 otherwise.

7. In $\mathbb{C}P^n$, let $U = \{ [x_0, \ldots, x_n] \mid x_0 \neq 0 \}$ and $V = \mathbb{C}P^n \setminus [1,0,\ldots,0]$. Use homotopies to show that $V$ has the same de Rham cohomology as $\mathbb{C}P^{n-1}$ and $U \cap V$ has the same de Rham cohomology as $S^{2n-1}$. Use this to show that $\dim H^k(\mathbb{C}P^n) = 1$ if $k$ is even and $0 \leq k \leq 2n$, 0 otherwise.