Mathematics G4403y
Modern Geometry

Assignment #9
Due February 19, 2014

1. Let M be an oriented Riemannian n-manifold. Show that there is a unique $\Omega \in \Omega^n(M)$, called the *volume form*, such that $\Omega(e_1, \ldots, e_n) = 1$ whenever $e_1, \ldots, e_n$ is an oriented orthonormal basis for any T_pM. Hint: show that, if x_i are any oriented coordinates, $\Omega = \sqrt{\det(g_{ij})}\, dx_1 \wedge \cdots \wedge dx_n$.

2. Let G be a Lie group, $\ell_g, r_g : G \to G$ left and right multiplication by g. A metric $\langle\, , \rangle$ is said to be *left-invariant* if for all $g \in G$, $\langle (\ell_g)_* v, (\ell_g)_* w\rangle = \langle v, w\rangle$, and *right-invariant* if a similar condition holds for r_g. It is *bi-invariant* if it is both left- and right-invariant. This exercise will show that a compact connected Lie group has a bi-invariant metric.

(a) Show that any Lie group G of dimension n has a left-invariant metric, and a nonzero left-invariant n-form ω: that is, one satisfying $\ell_g^* \omega = \omega$ for each g.

(b) If G is compact and connected, show that ω is also right-invariant. Hint: show first that $r_g^* \omega = f(g)\omega$ for some $f \in C^\infty(G)$. Then show that f is a homomorphism from G into the multiplicative group of $\mathbf{R}$. Then use compactness.

(c) Let $\langle\, , \rangle$ be a left-invariant metric on G compact and connected. Let ω be as in (b), with sign changed if necessary to ensure $\int_G \omega > 0$. Show that

$$\langle\langle u, v\rangle\rangle = \int_G \langle (r_g)_* u, (r_g)_* v\rangle\, \omega$$

defines a new, bi-invariant Riemannian metric on G.

3. Let M be a smooth manifold. If $\nabla^1, \ldots, \nabla^k$ are connections on TM and $\psi_1, \ldots, \psi_k \in C^\infty(M)$ satisfy $\sum_i \psi_i \equiv 1$, show that $\sum_i \psi_i \nabla^i$ is also a connection.

4. Let M be a smooth manifold, ∇ and $\tilde{\nabla}$ connections on TM.

(a) Show that the torsion $\tau_\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$ is a tensor field of type $(1, 2)$.

(b) Show that the difference $\nabla - \tilde{\nabla}$ is a tensor field of type $(1, 2)$.

(c) Conversely, if A is any tensor field of type $(1, 2)$, show that $\nabla + A$ is a connection.

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5. Let $H^n = \{\mathbf{x} \in \mathbf{R}^n \,|\, x_n > 0\}$ denote the n-dimensional upper half-space. For $c \in \mathbf{R}^+$ define a Riemannian metric on H^n by

$$g_{ij}(\mathbf{x}) = \frac{c}{x_n^2}\,\delta_{ij}.$$

(a) Calculate the Christoffel symbols of the Levi-Civita connection.

(b) Show that the geodesics are half-lines and semicircles that intersect the hyperplane $x_n = 0$ orthogonally, suitably parametrized. Hint: first do the case $n = 2$; for the general case show that you can change coordinates isometrically so that the initial tangent vector lies in the x_1, x_n-plane.

6. Let $\gamma : [a, b] \to M$ be a parametric curve on a Riemannian manifold. Assume for simplicity that $\gamma'(a) \neq 0$, $\gamma'(b) \neq 0$. Define the *parallel transport* $T_{\gamma(a)}M \to T_{\gamma(b)}M$ by taking a tangent vector at $\gamma(a)$, extending to a parallel vector field along γ, and evaluating at $\gamma(b)$. Show that parallel transport is linear, preserves the inner product, and preserves the orientation if M is oriented.

7. In Euclidean space, the parallel transport of a vector between two points does not depend on the choice of the curve between them. Give an example to show that this may be false on a general Riemannian manifold.