0. Show that in $\mathbb{R}^n$, a line (segment) is (up to reparametrization) the unique shortest (piecewise regular parametric) curve between two given endpoints.

1. Let $M$ be a connected Riemannian manifold. For $x, y \in M$, let $C(x, y)$ be the set of all piecewise regular parametric curves $\gamma$ from $x$ to $y$. Also let $\ell(\gamma)$ be the arclength of $\gamma$. Show that $d(x, y) := \inf_{\gamma \in C(x,y)} \ell(\gamma)$ defines a metric in the topological sense whose metric topology agrees with the usual topology on $M$. The hard part is to show that $d(x, y) > 0$ when $x \neq y$.

2. On a Riemannian manifold $M$, define the angle between tangent vectors $v, w \in T_p M$ to be $\arccos(\langle v, w \rangle / |v||w|)$. Show that a map of Riemannian manifolds is conformal if and only if it preserves angles. If $C$ is regarded as $\mathbb{R}^2$ with the standard metric, show that a map $C \to C$ is conformal if and only if it is holomorphic or antiholomorphic.

3. Show that the parametrization of a curve by arclength is canonical up to translations and reflections. That is, if $\gamma : [a, b] \to M$ is a regular parametric curve on a Riemannian manifold and $f : [c, d] \to [a, b]$ is a diffeomorphism, show that the arclength parametrizations of $\gamma$ and $\tilde{\gamma} = \gamma \circ f$ differ only by a reparametrization of the form $\tilde{s} = \pm s + k$ for some constant $k$.

4. If $\gamma(s)$ is a curve in $\mathbb{R}^n$ parametrized by arclength, recall that the unit tangent vector is $T = \gamma'(s)$ and the curvature is $\kappa(s) = |dT/ds|$. Show that if the curvature is zero, then $\gamma$ is a straight line. Two curves in $\mathbb{R}^n$ are congruent if there is a rigid motion $x \mapsto Ax + b$ of $\mathbb{R}^n$, for constant $A \in O(n), b \in \mathbb{R}^n$, taking one to the other. Give a counterexample in $\mathbb{R}^2$ to show that two curves with the same curvature need not be congruent. In $\mathbb{R}^2$, refine the definition of curvature and prove that with your refined definition, curves with the same curvature are congruent.

5. Show that any isometry from $\mathbb{R}^n$ to itself must take straight lines to straight lines. Show that the only such isometries are those of the form $x \mapsto Ax + b$ for constant $A \in O(n), b \in \mathbb{R}^n$.

6. Show that a sum of finitely many Riemannian metrics is a Riemannian metric. Use a partition of unity to show that every smooth manifold admits a Riemannian metric.

7. Find explicit formulas for the matrix elements $g_{ij}, 1 \leq i, j \leq 2$, of the induced metric on the 2-torus in $\mathbb{R}^3$ with coordinates $(\theta, \phi)$ embedded via

$$(\theta, \phi) \mapsto ((a + b \cos \phi) \cos \theta, (a + b \cos \phi) \sin \theta, b \sin \phi).$$

8. Given a Riemannian metric $g$ and a smooth vector field $X$ on a manifold $M$, define a Lie derivative $L_X g$ (assigning to each $p \in M$ a symmetric tensor $(L_X g)_p \in T^*_p M \otimes T^*_p M$) and show that the flow of $X$ acts by isometries if and only if $L_X g = 0$. 