Mathematics G4403y
Modern Geometry

Assignment #8
Due February 5, 2014

0. Show that in $\mathbf{R}^n$, a line (segment) is (up to reparametrization) the unique shortest (piecewise regular parametric) curve between two given endpoints.

1. Let M be a connected Riemannian manifold. For $x, y \in M$, let $C(x, y)$ be the set of all piecewise regular parametric curves γ from x to y. Also let $\ell(\gamma)$ be the arclength of γ. Show that $d(x, y) := \inf_{C(x,y)} \ell(\gamma)$ defines a metric in the topological sense whose metric topology agrees with the usual topology on M. The hard part is to show that $d(x, y) > 0$ when $x \neq y$.

2. On a Riemannian manifold M, define the *angle* between tangent vectors $v, w \in T_pM$ to be $\arccos(\langle v, w\rangle / |v||w|)$. Show that a map of Riemannian manifolds is conformal if and only if it preserves angles. If $\mathbf{C}$ is regarded as $\mathbf{R}^2$ with the standard metric, show that a map $\mathbf{C} \to \mathbf{C}$ is conformal if and only if it is holomorphic or antiholomorphic.

3. Show that the parametrization of a curve by arclength is canonical up to translations and reflections. That is, if $\gamma : [a, b] \to M$ is a regular parametric curve on a Riemannian manifold and $f : [c, d] \to [a, b]$ is a diffeomorphism, show that the arclength parametrizations of γ and $\tilde{\gamma} = \gamma \circ f$ differ only by a reparametrization of the form $\tilde{s} = \pm s + k$ for some constant k.

4. If $\gamma(s)$ is a curve in $\mathbf{R}^n$ parametrized by arclength, recall that the *unit tangent vector* is $T = \gamma'(s)$ and the curvature is $\kappa(s) = |dT/ds|$. Show that if the curvature is zero, then γ is a straight line. Two curves in $\mathbf{R}^n$ are *congruent* if there is a rigid motion $x \mapsto Ax + b$ of $\mathbf{R}^n$, for constant $A \in O(n)$, $b \in \mathbf{R}^n$, taking one to the other. Give a counterexample in $\mathbf{R}^2$ to show that two curves with the same curvature need not be congruent. In $\mathbf{R}^2$, refine the definition of curvature and prove that with your refined definition, curves with the same curvature are congruent.

5. Show that any isometry from $\mathbf{R}^n$ to itself must take straight lines to straight lines. Show that the only such isometries are those of the form $x \mapsto Ax + b$ for constant $A \in O(n)$, $b \in \mathbf{R}^n$.

6. Show that a sum of finitely many Riemannian metrics is a Riemannian metric. Use a partition of unity to show that every smooth manifold admits a Riemannian metric.

7. Find explicit formulas for the matrix elements g_{ij}, $1 \leq i, j \leq 2$, of the induced metric on the 2-torus in $\mathbf{R}^3$ with coordinates (θ, ϕ) embedded via

$$(\theta, \phi) \mapsto ((a + b\cos\phi)\cos\theta, (a + b\cos\phi)\sin\theta, b\sin\phi).$$

8. Given a Riemannian metric g and a smooth vector field X on a manifold M, define a Lie derivative $L_X g$ (assigning to each $p \in M$ a symmetric tensor $(L_X g)_p \in T_p^*M \otimes T_p^*M$) and show that the flow of X acts by isometries if and only if $L_X g = 0$.