General note: “computing” the de Rham cohomology means describing it up to isomorphism, which for a finite-dimensional vector space just means giving its dimension. So you just have to compute the $k$th Betti number $h^k(M) := \dim H^k(M)$ for each $k$.

There are 4 additional problems, which I’ll distribute in lecture on a handwritten sheet.

1. If $M$ is an oriented, connected, noncompact manifold of dimension $n$, then $H^n(M) = 0$.

2. (a) Compute the de Rham cohomology of $\mathbb{R}^n$ minus $m$ points. Use Mayer-Vietoris.
   
   (b) If $\ell \neq m$, show that $\mathbb{R}^n$ minus $\ell$ points is not diffeomorphic to $\mathbb{R}^n$ minus $m$ points.

3. (a) Find a diffeomorphism $\{u \in \mathbb{R}^2 \mid 0 < |u| < 3\} \to \{u \in \mathbb{R}^2 \mid 1 < |u| < 3\}$ which is the identity on $\{u \in \mathbb{R}^2 \mid 2 < |u| < 3\}$.
   
   (b) Let the trinion $T$ be the complement in $S^2$ of three disjoint embedded closed disks, say $x \geq 1 - \epsilon$, $y \geq 1 - \epsilon$, and $z \geq 1 - \epsilon$ for some small $\epsilon$. Show that the trinion is diffeomorphic to $\mathbb{R}^2$ minus 2 points. [Hint: define the map on each open set in a cover and invoke the gluing lemma.]
   
   (c) Compute $H^k(T)$ for each $k$, and the rank of the restriction maps $H^k(T) \to H^k(A)$, where $A$ is the disjoint union of the three annuli $1 - 2\epsilon < x < 1 - \epsilon$, $1 - 2\epsilon < y < 1 - \epsilon$, $1 - 2\epsilon < z < 1 - \epsilon$. Also compute $H^k_\omega(T)$ for each $k$.

4. Two compact, oriented, embedded submanifolds $N_0, N_1 \subset M$ are said to be cobordant if there is a compact, oriented, embedded submanifold $N \subset M \times [0, 1]$ with boundary $\partial N = -N_0 \times \{0\} \cup N_1 \times \{1\}$. Here $-N_0$ denotes $N_0$ with the opposite orientation. Show that if $N_0$ and $N_1$ are cobordant and $\nu \in Z^n(M)$, then $\int_{N_0} \nu = \int_{N_1} \nu$. Exhibit a cobordism on the trinion between a circle and the disjoint union of two others.

5. The Brouwer degree of a map. Let $f : M \to N$ be a smooth map between compact, connected, oriented manifolds of the same dimension $n$.

   (a) Show that $H^n(M) \cong \mathbb{R} \cong H^n(N)$.
   
   (b) Use Sard’s theorem to show that the set of regular values of $f$ is dense in $N$.
   
   (c) Show that the set of regular values of $f$ is open in $N$, and for any such value $y$, $f^{-1}(y)$ is finite.
   
   (d) For any $y \in N$, construct a closed form $\nu_y \in \Omega^n(N)$, supported in a small disk around $y$, so that $\int_N \nu_y = 1$. Show that $[\nu_y] = [\nu_z] \in H^n(N)$ for any $y, z \in N$.
   
   (e) For any regular point $x \in M$ of $f$, let $\text{sgn} x := \det D_x F / |\det D_x F|$, where $F = \psi \circ f \circ \phi^{-1}$ for oriented charts $\phi, \psi$ near $x$ and $f(x)$. Show that $\text{sgn} x$ does not depend on the choice of $\phi$ and $\psi$.
   
   (f) If $y \in N$ is a regular value of $f$, let $\text{deg}_y f := \sum_{x \in f^{-1}(y)} \text{sgn} x \in \mathbb{Z}$. Show that if $y, z$ are two regular values, then $\text{deg}_y f = \text{deg}_z f$. Call this the degree of $f$.
   
   (g) For each $n \in \mathbb{Z}$, give an example of a map $S^1 \to S^1$ of degree $n$.
   
   (h) If the degree of $f$ is nonzero, show that $f$ is surjective. Is the converse true?

6. (Extra credit) Identify the literary reference in I §3 of Bott & Tu.