Mathematics G4402x
Modern Geometry

Assignment #7

Due December 11, 2013

General note: "computing" the de Rham cohomology means describing it up to isomorphism, which for a finite-dimensional vector space just means giving its dimension. So you just have to compute the kth *Betti number* $h^k(M) := \dim H^k(M)$ for each k.

There are 4 additional problems, which I'll distribute in lecture on a handwritten sheet.

1. If M is an oriented, connected, noncompact manifold of dimension n, then $H^n(M) = 0$.

2. (a) Compute the de Rham cohomology of $\mathbb{R}^n$ minus m points. Use Mayer-Vietoris.

(b) If $\ell \neq m$, show that $\mathbb{R}^n$ minus ℓ points is not diffeomorphic to $\mathbb{R}^n$ minus m points.

3. (a) Find a diffeomorphism $\{\mathbf{u} \in \mathbb{R}^2 \,|\, 0 < |\mathbf{u}| < 3\} \to \{\mathbf{u} \in \mathbb{R}^2 \,|\, 1 < |\mathbf{u}| < 3\}$ which is the identity on $\{\mathbf{u} \in \mathbb{R}^2 \,|\, 2 < |\mathbf{u}| < 3\}$.

(b) Let the *trinion* T be the complement in S^2 of three disjoint embedded closed disks, say $x \geq 1 - \epsilon$, $y \geq 1 - \epsilon$, and $z \geq 1 - \epsilon$ for some small ϵ. Show that the trinion is diffeomorphic to $\mathbb{R}^2$ minus 2 points. [Hint: define the map on each open set in a cover and invoke the gluing lemma.]

(c) Compute $H^k(T)$ for each k, and the rank of the restriction maps $H^k(T) \to H^k(A)$, where A is the disjoint union of the three annuli $1 - 2\epsilon < x < 1 - \epsilon$, $1 - 2\epsilon < y < 1 - \epsilon$, $1 - 2\epsilon < z < 1 - \epsilon$. Also compute $H^k_c(T)$ for each k.

4. Two compact, oriented, embedded submanifolds N_0, $N_1 \subset M$ are said to be *cobordant* if there is a compact, oriented, embedded submanifold $N \subset M \times [0, 1]$ with boundary $\partial N = -N_0 \times \{0\} \cup N_1 \times \{1\}$. Here $-N_0$ denotes N_0 with the opposite orientation. Show that if N_0 and N_1 are cobordant and $\nu \in Z^n(M)$, then $\int_{N_0} \nu = \int_{N_1} \nu$. Exhibit a cobordism on the trinion between a circle and the disjoint union of two others.

5. *The Brouwer degree of a map.* Let $f : M \to N$ be a smooth map between compact, connected, oriented manifolds of the same dimension n.

(a) Show that $H^n(M) \cong \mathbb{R} \cong H^n(N)$.

(b) Use Sard's theorem to show that the set of regular values of f is dense in N.

(c) Show that the set of regular values of f is open in N, and for any such value y, $f^{-1}(y)$ is finite.

(d) For any $y \in N$, construct a closed form $\nu_y \in \Omega^n(N)$, supported in a small disk around y, so that $\int_N \nu_y = 1$. Show that $[\nu_y] = [\nu_z] \in H^n(N)$ for any $y, z \in N$.

(e) For any regular point $x \in M$ of f, let $\operatorname{sgn} x := \det D_x F / |\det D_x F|$, where $F = \psi \circ f \circ \phi^{-1}$ for oriented charts ϕ, ψ near x and $f(x)$. Show that $\operatorname{sgn} x$ does not depend on the choice of ϕ and ψ.

(f) If $y \in N$ is a regular value of f, let $\deg_y f := \sum_{x \in f^{-1}(y)} \operatorname{sgn} x \in \mathbb{Z}$. Show that if y, z are two regular values, then $\deg_y f = \deg_z f$. Call this the *degree* of f.

(g) For each $n \in \mathbb{Z}$, give an example of a map $S^1 \to S^1$ of degree n.

(h) If the degree of f is nonzero, show that f is surjective. Is the converse true?

6. (Extra credit) Identify the literary reference in I §3 of Bott & Tu.