# Mathematics G4402x
# Modern Geometry

### Assignment #6

Due November 25, 2013

**1.** (a) An isomorphism of oriented vector spaces is said to be *orientation-preserving* if it takes oriented bases to oriented bases, and *orientation-reversing* otherwise. If $F : M \to N$ is a diffeomorphism of connected manifolds, show that $D_pF$ is either everywhere orientation-preserving or everywhere orientation-reversing. (If the former, $F$ itself is said to be orientation-preserving.)

(b) For which $n$ is the antipodal map $F : S^n \to S^n$ given by $F(v) = -v$ orientation-preserving?

**2.** Let a finite group $\Gamma$ act smoothly and freely on a connected oriented manifold $M$.

(a) Show that the pullback induces an isomorphism

$$\Omega^k(M/\Gamma) \cong \{\nu \in \Omega^k(M) \,|\, \gamma^*\nu = \nu \text{ for all } \gamma \in \Gamma\}.$$

(b) Show that $M/\Gamma$ is orientable if and only if the action of every $\gamma \in \Gamma$ is orientation-preserving.

(c) Use the quotient map from $S^n$ to show that $\mathbb{RP}^n$ is orientable if and only if $n$ is odd.

**3.** If $M$ is a manifold and 0 is a regular value of $f \in C^\infty(M)$, show that $f^{-1}[0, \infty)$ is a manifold with boundary.

**4.** If $\mu$ and $\nu$ are closed forms, show that $\mu \wedge \nu$ is closed. If, in addition, $\nu$ is exact, show that $\mu \wedge \nu$ is exact.

**5.** Sketch how the ordinary Divergence Theorem in $\mathbb{R}^3$ may be obtained as a special case of the general Stokes's Theorem.

**6.** Let $\omega \in \Omega^{n-1}(\mathbb{R}^n)$. Prove that $\omega$ is closed if and only if its integral over every $(n-1)$-sphere is zero (regardless of the center and radius).

**7.** Let $\nu \in \Omega^1(\mathbb{R}^2 \setminus (0, 0))$ be the form "$d\theta$" discussed in class, namely

$$\nu = \frac{x\,dy - y\,dx}{x^2 + y^2}.$$

Determine $\int_S \nu$ and $\int_T \nu$, where $S$ and $T$ are circles of radius 5, oriented counterclockwise, with centers at $(1, 2)$ and $(7, 2)$ respectively. Explain your reasoning.

**8.** (Only for those who have never studied differential forms before.) Make sure you know how to integrate differential forms in practice by parametrizing a specific, nontrivial 1-manifold with boundary (e.g. a helix) in $\mathbb{R}^3$ and integrating a specific, nontrivial $\nu \in \Omega^1(\mathbb{R}^3)$ over it. Same for a 2-manifold with boundary in $\mathbb{R}^3$ (e.g. a hemisphere) and a 2-form.