1. (a) An isomorphism of oriented vector spaces is said to be orientation-preserving if it takes oriented bases to oriented bases, and orientation-reversing otherwise. If $F : M \rightarrow N$ is a diffeomorphism of connected manifolds, show that $D_p F$ is either everywhere orientation-preserving or everywhere orientation-reversing. (If the former, $F$ itself is said to be orientation-preserving.)

(b) For which $n$ is the antipodal map $F : S^n \rightarrow S^n$ given by $F(v) = -v$ orientation-preserving?

2. Let a finite group $\Gamma$ act smoothly and freely on a connected oriented manifold $M$.

(a) Show that the pullback induces an isomorphism
\[ \Omega^k(M/\Gamma) \cong \{ \nu \in \Omega^k(M) \mid \gamma^* \nu = \nu \text{ for all } \gamma \in \Gamma \}. \]

(b) Show that $M/\Gamma$ is orientable if and only if the action of every $\gamma \in \Gamma$ is orientation-preserving.

(c) Use the quotient map from $S^n$ to show that $\mathbb{RP}^n$ is orientable if and only if $n$ is odd.

3. If $M$ is a manifold and 0 is a regular value of $f \in C^\infty(M)$, show that $f^{-1}[0, \infty)$ is a manifold with boundary.

4. If $\mu$ and $\nu$ are closed forms, show that $\mu \wedge \nu$ is closed. If, in addition, $\nu$ is exact, show that $\mu \wedge \nu$ is exact.

5. Sketch how the ordinary Divergence Theorem in $\mathbb{R}^3$ may be obtained as a special case of the general Stokes’s Theorem.

6. Let $\omega \in \Omega^{n-1}(\mathbb{R}^n)$. Prove that $\omega$ is closed if and only if its integral over every $(n-1)$-sphere is zero (regardless of the center and radius).

7. Let $\nu \in \Omega^1(\mathbb{R}^2 \setminus (0,0))$ be the form “$d\theta$” discussed in class, namely
\[ \nu = \frac{x \, dy - y \, dx}{x^2 + y^2}. \]

Determine $\int_S \nu$ and $\int_T \nu$, where $S$ and $T$ are circles of radius 5, oriented counterclockwise, with centers at $(1,2)$ and $(7,2)$ respectively. Explain your reasoning.

8. (Only for those who have never studied differential forms before.) Make sure you know how to integrate differential forms in practice by parametrizing a specific, nontrivial 1-manifold with boundary (e.g. a helix) in $\mathbb{R}^3$ and integrating a specific, nontrivial $\nu \in \Omega^1(\mathbb{R}^3)$ over it. Same for a 2-manifold with boundary in $\mathbb{R}^3$ (e.g. a hemisphere) and a 2-form.